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Research Paper

Equation of Geodesic - Perihelion of Mercury and Other Planets Using Isotropic Line Element

Physics

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ABSTRACT

Einstein's law of geodesic motion is applied to the planet Mercury using anisotropic line element. The potential given by anisotropic line element at a point outside the source does not have Newtonian approximation. Eddington has given the condition for Newtonian approximation that the line element given by $-(1+2\Omega)(dx^2+dy^2+dz^2)+(1-2\Omega)dt^2=ds^2$ *gives Newtonian approximation of field only if is independent of time. This condition on is satisfied by the orbits because the field outside the source is independent of time.*

The radius of the orbit of Mercury cm and is fairly large but the departure from Newton's theory of the orbit is also too small so that second order terms in perturbation have to be considered to explain motion of perihelion. So the difference between isotropic and anisotropic metric in the calculation of perihelion may be enough significant. With this point of view we have tried to study the same problem of perihelion of Mercury and other planets using isotropic coordinate system.

Introduction: A physical definition of a geodesic is that of a curve along which a freely falling particle moves. The path of the particle is determined by equation of geodesic. The gravitational mass m is the measure adopted in Newton's theory of the power of the particle in causing a field of acceleration around it; the units being here chosen are gravitational units i.e. $G=c=1$with this mass of the sun is approximately 1.5 Km. Planetary orbits can be studied using geodesic equation. In the present investigation isotropic line element is considered.

Equation of geodesic using isotropic line element:

The track of a particle moving freely in the space-time given by

$$ds^2 = -e^{\mu}\left(dr^2 + r^2d\theta^2 + r^2\mathrm{Sin}^2\theta\cdot d\phi^2\right) + e^{\nu}dt^2 \text{ ---------- (1)}$$

is determined by the equations of geodesic[1]

$$\frac{d^2x^{\alpha}}{ds^2} + \{\mu\nu,\alpha\}\frac{dx^{\mu}}{ds}\cdot\frac{dx^{\nu}}{ds} = 0 \text{ ---------- (2)}$$

Taking $\alpha = 2$ we get

$$\frac{d^2x^{2}}{ds^2} + \{\mu\nu,2\}\frac{dx^{\mu}}{ds}\cdot\frac{dx^{\nu}}{ds} = 0$$

$$\frac{d^2\theta}{ds^2} + \{\mu\nu,2\}\frac{dx^{\mu}}{ds}\cdot\frac{dx^{\nu}}{ds} = 0 \because x^2 = \theta$$

$$\frac{d^2\theta}{ds^2} + \{12,2\}\frac{dx^{1}}{ds}\cdot\frac{dx^{2}}{ds} + \{21,2\}\frac{dx^{2}}{ds}\frac{dx^{1}}{ds} + \{24,2\}\frac{dx^{2}}{ds}\frac{dx^{4}}{ds}$$

$$+ \{33,2\}\frac{dx^{3}}{ds}\frac{dx^{3}}{ds} + \{42,2\}\frac{dx^{4}}{ds}\frac{dx^{2}}{ds} = 0$$

$$\frac{d^2\theta}{ds^2} + 2\{12,2\}\frac{dr}{ds}\cdot\frac{d\theta}{ds} + 2\{24,2\}\frac{d\theta}{ds}\frac{dt}{ds} + \{33,2\}\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} + 2\left(\frac{1}{r} + \frac{\mu'}{2}\right)\frac{dr}{ds}\cdot\frac{d\theta}{ds} + 2\left(\frac{\dot{\mu}}{2}\right)\frac{d\theta}{ds}\frac{dt}{ds} - \text{Sin}\theta\cdot\text{Cos}\theta\cdot\left(\frac{d\phi}{ds}\right)^2 = 0$$

Initial conditions: Let us choose the coordinates so that the particle moves initially in the plane $\theta = \frac{\pi}{2}$ then $\frac{d\theta}{ds} = 0$ initially and $\text{Cos}\theta = 0$ so that $\frac{d^2\theta}{ds^2} = 0$ particle therefore continues to move in this plane.

Equations for $\alpha = 3,4,1$ are as follows

For $\alpha = 3$

$$\frac{d^2x^3}{ds^2} + \{\mu\nu,3\}\frac{dx^\mu}{ds}\cdot\frac{dx^\nu}{ds} = 0$$

$$\frac{d^2\phi}{ds^2} + \{\mu\nu,3\}\frac{dx^\mu}{ds}\cdot\frac{dx^\nu}{ds} = 0 \because x^3 = \phi$$

$$\frac{d^2\phi}{ds^2} + \{13,3\}\frac{dx^1}{ds}\cdot\frac{dx^3}{ds} + \{31,3\}\frac{dx^3}{ds}\cdot\frac{dx^1}{ds}$$

$$+ \{23,3\}\frac{dx^2}{ds}\cdot\frac{dx^3}{ds} + \{32,3\}\frac{dx^3}{ds}\cdot\frac{dx^2}{ds}$$

$$+ \{34,3\}\frac{dx^3}{ds}\cdot\frac{dx^4}{ds} + \{43,3\}\frac{dx^4}{ds}\cdot\frac{dx^3}{ds} = 0$$

$$\frac{d^2\phi}{ds^2} + 2\{13,3\}\frac{dr}{ds}\cdot\frac{d\phi}{ds} + 2\{23,3\}\frac{d\theta}{ds}\cdot\frac{d\phi}{ds} + 2\{34,3\}\frac{d\phi}{ds}\cdot\frac{dt}{ds} = 0$$

$$\frac{d^2\phi}{ds^2} + 2\left(\frac{1}{r} + \frac{\mu'}{2}\right)\frac{dr}{ds}\cdot\frac{d\phi}{ds} + 2\text{Cot}\theta\frac{d\theta}{ds}\cdot\frac{d\phi}{ds} + 2\left(\frac{\dot{\mu}}{2}\right)\frac{d\phi}{ds}\cdot\frac{dt}{ds} = 0$$

Since exterior solution is independent of time $\dot{\mu} = 0$

$$\therefore \frac{d^2\phi}{ds^2} + 2\left(\frac{1}{r} + \frac{\mu'}{2}\right)\frac{dr}{ds}\cdot\frac{d\phi}{ds} = 0$$

$$\therefore \frac{d^2\phi}{ds^2} + \left(\frac{2}{r} + \mu'\right)\frac{dr}{ds}\cdot\frac{d\phi}{ds} = 0$$

$$\int\frac{\left(\frac{d^2\phi}{ds^2}\right)}{\left(\frac{d\phi}{ds}\right)}\cdot ds = \int -\left(\frac{2}{r} + \mu'\right)\frac{dr}{ds}\cdot ds$$

$$\text{Log}\left(\frac{d\phi}{ds}\right) = -2\text{Log}r - \mu + C_1$$

$$\frac{d\phi}{ds} = C_1\frac{e^{-\mu}}{r^2} \text{ where } C_1 \text{ is constant.}$$

$$\frac{d\phi}{ds} = h\frac{e^{-\mu}}{r^2} \quad \text{----------} \quad (3)$$

For $\alpha = 4$

$$\frac{d^2x^4}{ds^2} + \{\mu\nu,4\}\frac{dx^\mu}{ds}\cdot\frac{dx^\nu}{ds} = 0$$

$$\frac{d^2t}{ds^2} + \{\mu\nu,4\}\frac{dx^\mu}{ds}\cdot\frac{dx^\nu}{ds} = 0 \quad \because x^4 = t$$

$$\frac{d^2t}{ds^2} + \{114\}\frac{dx^1}{ds}\cdot\frac{dx^1}{ds} + \{224\}\frac{dx^2}{ds}\cdot\frac{dx^2}{ds} + 2\{144\}\frac{dx^1}{ds}\cdot\frac{dx^4}{ds}$$

$$+ \{33,4\}\frac{dx^3}{ds}\cdot\frac{dx^3}{ds} + \{44,4\}\frac{dx^4}{ds}\cdot\frac{dx^4}{ds} = 0$$

$$\therefore \frac{d^2t}{ds^2} + 2\{14,4\}\frac{dx^1}{ds}\cdot\frac{dx^4}{ds} = 0$$

$$\therefore \frac{d^2t}{ds^2} + 2\left\{\frac{\nu'}{2}\right\}\frac{dr}{ds}\cdot\frac{dt}{ds} = 0$$

$$\therefore \frac{d^2t}{ds^2} + \nu'\frac{dr}{ds}\cdot\frac{dt}{ds} = 0$$

$$\frac{\left(\frac{d^2t}{ds^2}\right)}{\left(\frac{dt}{ds}\right)} = -\nu' \frac{dr}{ds}$$

$$\therefore \text{Log}\frac{dt}{ds} = -\nu + C_2$$

$$\therefore \frac{dt}{ds} = C \cdot e^{-\nu} \text{ ---------- (4)}$$

$$\frac{d\theta}{ds} = 0$$

$$\frac{d\phi}{ds} = h\frac{e^{-\mu}}{r^2}$$

$$\frac{dt}{ds} = C \cdot e^{-\nu} = \frac{C}{e^{\nu}} = \frac{C}{\gamma}$$

e^{μ} and e^{ν} are to be taken from exterior solution given by Wyman[2]

$$-e^{\mu}\left(dr^2 + r^2d\theta^2 + r^2\text{Sin}^2\theta \cdot d\phi^2\right) + e^{\nu}dt^2 = ds^2$$

$$\therefore -e^{\mu}\left[\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2\text{Sin}^2\theta \cdot \left(\frac{d\phi}{ds}\right)^2\right] + e^{\nu}\left(\frac{dt}{ds}\right)^2 = 1$$

$$\therefore -e^{\mu}\left[\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2\right] + e^{\nu}\left(\frac{dt}{ds}\right)^2 = 1$$

$$\therefore -e^{\mu}\left[\left(\frac{dr}{d\phi} \cdot \frac{d\phi}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2\right] + e^{\nu}\left(\frac{dt}{ds}\right)^2 = 1$$

But $\frac{d\phi}{ds} = h\frac{e^{-\mu}}{r^2}$ and $\frac{dt}{ds} = C \cdot e^{-\nu}$

$$\therefore -e^{\mu}\left[\left(\frac{dr}{d\phi} \cdot h\frac{e^{-\mu}}{r^2}\right)^2 + r^2\left(h\frac{e^{-\mu}}{r^2}\right)^2\right] + e^{\nu}\left(C \cdot e^{-\nu}\right)^2 = 1$$

$$\therefore \left[\left(\frac{dr}{d\phi}\right)^2 \cdot h^2\frac{1}{r^4} + \frac{h^2}{r^2}\right] = C^2 \cdot e^{\mu-\nu} - e^{\mu}$$

According to Wyman

$$e^{\mu} = q^2 = \left(1 + \frac{m}{2r}\right)^4 \text{ ---------- (5)} \qquad \text{where } q = 1 + \frac{m}{2r}$$

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$$e^{\nu} = \frac{\left(q - \frac{m}{r}\right)^2}{q^2} \quad \text{---------- (6)}$$

$$e^{\nu} = \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2} \therefore e^{-\nu} = \frac{\left(1 + \frac{m}{2r}\right)^2}{\left(1 - \frac{m}{2r}\right)^2}$$

$$\therefore \left[\left(\frac{dr}{d\phi}\right)^2 \cdot h^2 \frac{1}{r^4} + \frac{h^2}{r^2}\right] = C^2 \cdot \left(1 + \frac{m}{2r}\right)^4 \cdot \frac{\left(1 + \frac{m}{2r}\right)^2}{\left(1 - \frac{m}{2r}\right)^2} - \left(1 + \frac{m}{2r}\right)^4$$

$$\therefore \left[\left(\frac{dr}{d\phi}\right)^2 \cdot h^2 \frac{1}{r^4} + \frac{h^2}{r^2}\right] = C^2 \cdot \frac{\left(1 + \frac{m}{2r}\right)^6}{\left(1 - \frac{m}{2r}\right)^2} - \left(1 + \frac{m}{2r}\right)^4 \text{---------- (7)}$$

Let $\alpha = \frac{m}{2r}$ then with these values equation (7) becomes

$$\therefore \left[\left(\frac{dr}{d\phi}\right)^2 \cdot h^2 \frac{1}{r^4} + \frac{h^2}{r^2}\right] = C^2 \frac{\left(1 + 6\alpha + 15\alpha^2 + 20\alpha^3 + 15\alpha^4\right)}{\left(1 - 2\alpha + \alpha^2\right)}$$

$$- \left(1 + 4\alpha + 6\alpha^2 + 4\alpha^3 + \alpha^4\right)$$

$$\frac{\left(1 + 6\alpha + 15\alpha^2 + 20\alpha^3 + 15\alpha^4\right)}{\left(1 - 2\alpha + \alpha^2\right)} \approx 1 + 8\alpha + 30\alpha^2 + 72\alpha^3$$

$$\therefore \left[\left(\frac{dr}{d\phi}\right)^2 \cdot h^2 \frac{1}{r^4} + \frac{h^2}{r^2}\right] = C^2\left(1 + 8\alpha + 30\alpha^2 + 72\alpha^3\right)$$

$$- \left(1 + 4\alpha + 6\alpha^2 + 4\alpha^3 + \alpha^4\right)$$

Putting $\alpha = \frac{m}{2r}$ we get

$$\therefore \left[\left(\frac{dr}{d\phi}\right)^2 \cdot h^2 \frac{1}{r^4} + \frac{h^2}{r^2}\right] = C^2\left[1 + 8\frac{m}{2r} + 30\left(\frac{m}{2r}\right)^2 + 72\left(\frac{m}{2r}\right)^3\right]$$

$$- \left[1 + 4\frac{m}{2r} + 6\left(\frac{m}{2r}\right)^2 + 4\left(\frac{m}{2r}\right)^3\right]$$

$$\therefore \left[\left(\frac{dr}{d\phi}\right)^2 \cdot h^2 \frac{1}{r^4} + \frac{h^2}{r^2}\right] = C^2\left[1 + 4\frac{m}{r} + \frac{15}{2}\frac{m^2}{r^2} + 9\frac{m^3}{r^3}\right]$$

$$-\left[1 + 2\frac{m}{r} + \frac{3}{2}\frac{m^2}{r^2} + \frac{1}{2}\frac{m^3}{r^3}\right]$$

Now writing $u = \frac{1}{r} \therefore du = -\frac{1}{r^2} dr \therefore -\frac{1}{u^2} du = dr \therefore -\frac{1}{u^2}\frac{du}{d\phi} = \frac{dr}{d\phi}$

$$\therefore \left[\left(-\frac{1}{u^2}\frac{du}{d\phi}\right)^2 \cdot h^2u^4 + h^2u^2\right] = C^2\left[1 + 4mu + \frac{15}{2}m^2u^2 + 9m^3u^3\right]$$

$$-\left[1 + 2mu + \frac{3}{2}m^2u^2 + \frac{1}{2}m^3u^3\right]$$

$$\therefore \left[\left(\frac{du}{d\phi}\right)^2 \cdot h^2 + h^2u^2\right] = C^2 - 1 + 4muC^2 + \frac{15}{2}m^2u^2C^2 + 9m^3u^3C^2$$

$$-2mu - \frac{3}{2}m^2u^2 - \frac{m^3u^3}{2}$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{C^2 - 1}{h^2} + \frac{4muC^2 + \frac{15}{2}m^2u^2C^2 + 9m^3u^3C^2 - 2mu - \frac{3}{2}m^2u^2 - \frac{m^3u^3}{2}}{h^2}$$

-------- (8)

The solution of $\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2}$ is

$u = \frac{m}{h^2}\left[1 + e \cdot \text{Cos}(\phi - \omega)\right]$ where e is eccentricity of the planet.

Let us put $\omega = 0$ for simplicity.

$u = \frac{m}{h^2}\left[1 + e \cdot \text{Cos}\phi\right]$---------- (9)

$$\therefore \frac{du}{d\phi} = \frac{m}{h^2}(-e\text{Sin}\phi)$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 = \left(\frac{m}{h^2}\right)^2 e^2\text{Sin}^2\phi$$

and $u^2 = \left(\frac{m}{h^2}\right)^2\left[1 + e \cdot \text{Cos}\phi\right]^2$

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$$\therefore \left(\frac{du}{d\phi}\right)^2 + u^2 = \left(\frac{m}{h^2}\right)^2 e^2 Sin^2\phi + \left(\frac{m}{h^2}\right)^2 [1 + e \cdot Cos\phi]^2$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 + u^2 = \left(\frac{m}{h^2}\right)^2 (1 + 2 \cdot e \cdot Cos\phi + e^2)$$

Compare this equation with the following equation

$$\therefore \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{C^2 - 1}{h^2}$$

$$+ \frac{4muC^2 + \frac{15}{2} m^2u^2C^2 + 9m^3u^3C^2 - 2mu - \frac{3}{2} m^2u^2 - \frac{m^3u^3}{2}}{h^2}$$

Therefore one can see that

$$\frac{m^2}{h^4}(1 + e^2) = \frac{C^2 - 1}{h^2}$$

$$\therefore \frac{m^2}{h^2}(1 + e^2) = C^2 - 1$$

$$\therefore C^2 = 1 + \frac{m^2}{h^2}(1 + e^2)$$

$\because h^2 = m\ell$ we get

$$C^2 = 1 + \frac{m}{\ell}(1 + e^2) \quad \text{---------- (10)}$$

According to Eddington pp 88

$$h^2 = m\ell = ma(1 - e^2)$$

$$\therefore \ell = a(1 - e^2) \quad \text{---------- (11)}$$

Here ‘a’ is length of semi major axis. The length of semi minor axis ‘b’ is given by

$$b = a\sqrt{1 - e^2}$$

Planet	a Semi major Axis (10^6 Km)	e Orbital Eccentricity	Semi Latus Rectum (10^6 Km)	Precession per revolution (arc Sec)	Revolutions per century	Precession per century (arc Sec)
Mercury	57.9000	0.2060	55.4430	0.1034	414.9378	42.9195
Venus	108.2000	0.0070	108.1947	0.0530	162.6016	8.6186
Earth	149.6000	0.0170	149.5568	0.0383	100.0000	3.8345
Mars	227.9000	0.0930	225.9289	0.0254	53.1915	1.3502
Jupiter	778.3000	0.0480	776.5068	0.0074	8.4317	0.0623
Saturn	1427.0000	0.0560	1422.5249	0.0040	3.3944	0.0137
Uranus	2869.6000	0.0470	2863.2611	0.0020	1.1903	0.0024
Neptune	4496.6000	0.0090	4496.2358	0.0013	0.6068	0.0008
Pluto	5900.0000	0.2500	5531.2500	0.0010	0.4032	0.0004

m = 1.475 Kilometres - mass of the Sun

$C^2 = 1 + \frac{m}{\ell}(1 + e^2)$ The values of C^2 and ℓ for different planets are

Planet	a (10^6 Km)	$\ell = a(1-e^2)$	$C^2 = 1 + \frac{m}{\ell}(1+e^2)$
Mercury	57.9000	54.5811×10^{11}	1.000000028
Venus	108.2000	108.1947×10^{11}	1.000000013
Earth	149.6000	149.6×10^{11}	1.000000009
Mars	227.9000	225.9289×10^{11}	1.000000006
Jupiter	778.3000	776.5068×10^{11}	1.000000001
Saturn	1427.0000	1422.5249×10^{11}	1.000000001
Uranus	2869.6000	2863.261×10^{11}	$1+5.1453473 \times 10^{-10}$

Now, let us consider equation

$$\therefore \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{C^2 - 1}{h^2} + \frac{4muC^2 + \frac{15}{2}m^2u^2C^2 + 9m^3u^3C^2 - 2mu - \frac{3}{2}m^2u^2 - \frac{m^3u^3}{2}}{h^2}$$

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Looking at the values of C^2 in the table above if we assume $C^2 \approx 1$ then

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{4mu + \frac{15}{2}m^2u^2 + 9m^3u^3 - 2mu - \frac{3}{2}m^2u^2 - \frac{m^3u^3}{2}}{h^2}$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{1}{h^2}\left(2mu + 6m^2u^2 + \frac{17}{2}m^3u^3\right)$$

Differentiating w.r.t ϕ we get

$$2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + 2u\frac{du}{d\phi} = \frac{1}{h^2}\left(2m\frac{du}{d\phi} + 6.2.u.m^2\frac{du}{d\phi} + \frac{17}{2}\cdot 3m^3u^2\frac{du}{d\phi}\right)$$

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{h^2}\left(m + 6.m^2u + \frac{51}{4}m^3u^2\right)$$

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + \frac{6.m^2u}{h^2} + \frac{51}{4}\frac{m^3u^2}{h^2} \text{ ---------- (12)}$$

$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2}$ is Newtonian equation of the orbit whose solution is

$u = \frac{m}{h^2}\left[1 + e\cdot Cos(\phi - \omega)\right]$ where e is eccentricity of the planet.

Put $\omega = 0$ as it gives only the orientation of the orbit.

$$u = \frac{m}{h^2}\left[1 + e\cdot Cos\phi\right]$$

$\frac{6.m^2u}{h^2} + \frac{51}{4}\frac{m^3u^2}{h^2}$ is the non-Newtonian term in equation (12)

$$\frac{d^2u}{d\phi^2} + u\left(1 - \frac{6.m^2}{h^2}\right) = \frac{m}{h^2} + \frac{51}{4}\frac{m^3u^2}{h^2} \text{ ---------- (13)}$$

$$\frac{d^2u}{d\phi^2} + \alpha^2u = \frac{m}{h^2} \text{ --- (13)}$$

where $\alpha^2 = \left(1 - \frac{6.m^2}{h^2}\right)$ and the value of $\frac{51}{4}\frac{m^3u^2}{h^2}$ is of the order of 10^{-24} is

treated as a perturbation.

Auxiliary equation from which complimentary function is to be found out is

$$\frac{d^2u}{d\phi^2} + \alpha^2u = 0$$

This gives C.F as $u_c = ASin\alpha\phi + BCos\alpha\phi$ ------ (14)

The particular integral is to be found out by the following method

$$\left(D^2 + \alpha^2\right)u_p = \frac{m}{h^2}$$

$$\left(D + i\alpha\right)u_p = \frac{1}{\left(D - i\alpha\right)}\frac{m}{h^2}$$

$$\left(D + i\alpha\right)u_p = -\frac{me^{i\alpha\phi}}{i\alpha h^2}\int e^{-i\alpha\phi}d\left(-i\alpha\phi\right)$$

$$\left(D + i\alpha\right)u_p = \frac{im}{\alpha h^2}$$

$$u_p = \frac{im}{\alpha h^2}\frac{1}{\left(D + i\alpha\right)}$$

$$u_p = \frac{m}{\alpha^2 h^2}e^{-i\alpha\phi}\int e^{i\alpha\phi}d\left(i\alpha\phi\right)$$

$$u_p = \frac{m}{\alpha^2 h^2} \quad \text{---------- (15)}$$

Therefore the general solution is

$$u = u_c + u_p$$

$$\therefore u = A\mathrm{Sin}\alpha\phi + B\mathrm{Cos}\alpha\phi + \frac{m}{\alpha^2 h^2} \quad \text{---------- (16)}$$

For the sake of simplicity we shift the direction of polar axis in such a way that r is minimal i.e. planet is closest to the origin when $\phi = 0$ This means that u is to be maximum in this direction so $\frac{du}{d\phi} = 0$ and $\frac{d^2u}{d\phi^2} < 0$ when $\phi = 0$

$$\therefore \frac{1}{r} = B\mathrm{Cos}\alpha\phi + \frac{m}{\alpha^2 h^2}$$

$$\therefore r = \frac{1}{B\mathrm{Cos}\alpha\phi + \frac{m}{\alpha^2 h^2}}$$

$$\therefore r = \frac{\frac{\alpha^2 h^2}{m}}{1 + \frac{B\alpha^2 h^2}{m}\mathrm{Cos}\alpha\phi}$$

$$\therefore r = \frac{\frac{\alpha^2 h^2}{m}}{1 + e\mathrm{Cos}\alpha\phi} \text{------ (17) where we have put } e = \frac{B\alpha^2 h^2}{m}$$

$$\frac{d^2u}{d\phi^2} + \alpha^2 u = \frac{m}{h^2} + \frac{51}{4}\frac{m^3u^2}{h^2}$$

Neglecting the small term $\frac{51}{4}\frac{m^3u^2}{h^2}$ the solution is

$$\therefore u = \frac{m}{\alpha^2 h^2}\left[1 + eCos(\alpha\phi - \omega)\right]$$

Substitute this first approximation in the small term of

$$\frac{d^2u}{d\phi^2} + \alpha^2 u = \frac{m}{h^2} + \frac{51}{4}\frac{m^3u^2}{h^2}$$

$$\frac{d^2u}{d\phi^2} + \alpha^2 u = \frac{m}{h^2} + \frac{51}{4}\frac{m^3}{h^2}\left\{\frac{m}{\alpha^2 h^2}\left[1 + eCos(\alpha\phi - \omega)\right]\right\}^2$$

$$= \frac{m}{h^2} + \frac{51}{4}\frac{m^3}{h^2}\left\{\frac{m^2}{\alpha^4 h^4}\left[1 + 2eCos(\alpha\phi - \omega) + e^2Cos^2(\alpha\phi - \omega)\right]\right\}$$

Of the additional terms the only one term which can produce effect within the range of observation is the term containing $Cos(\alpha\phi - \omega)$

The particular integral of

$$\frac{d^2u}{d\phi^2} + \alpha^2 u = A\alpha^2 Cos(\alpha\phi - \omega) \text{is}$$

$u = \frac{1}{2}A\alpha\phi Sin(\alpha\phi - \omega)$ This term is a part of u

$u_1 = \frac{51}{4}\frac{m^5}{\alpha^4 h^6}e\alpha\phi Sin(\alpha\phi - \omega)$ which must be added to the complementary integral.

Thus the second approximation is

$$\therefore u = \frac{m}{\alpha^2 h^2}\left[1 + eCos(\alpha\phi - \omega)\right] + \frac{51}{4}\frac{m^5}{\alpha^4 h^6}e\alpha\phi Sin(\alpha\phi - \omega)$$

$$u = \frac{m}{\alpha^2 h^2}\left\{1 + eCos(\alpha\phi - \omega) + \frac{51}{4}\frac{m^4}{\alpha h^4}e\phi Sin(\alpha\phi - \omega)\right\}$$

$$u = \frac{m}{\alpha^2 h^2}\left\{1 + e\left[Cos(\alpha\phi - \omega) + \left(\frac{51}{4}\frac{m^4}{\alpha h^4}\phi\right)Sin(\alpha\phi - \omega)\right]\right\}$$

$$u = \frac{m}{\alpha^2 h^2}\left\{1 + e\left[Cos\delta\omega Cos(\alpha\phi - \omega) + Sin\delta\omega Sin(\alpha\phi - \omega)\right]\right\}$$

Where $Sin\delta\omega = \left(\frac{51}{4}\frac{m^4}{\alpha h^4}\phi\right)$ and $Cos\delta\omega = 1$

$$u = \frac{m}{\alpha^2 h^2}\{1 + eCos(\alpha\phi - \omega - \delta\omega)\} \text{ ---------- (18)}$$

$$\text{Where } Sin\delta\omega = \delta\omega = \left(\frac{51}{4}\frac{m^4}{\alpha h^4}\phi\right)$$

$$\frac{\delta\omega}{\phi} = \frac{51}{4}\frac{m^4}{\alpha h^4}$$

$$\frac{\delta\omega}{\phi} = \frac{51}{4}\frac{m^4}{\alpha m^2 \ell^2} \because h^2 = m\ell$$

$$\frac{\delta\omega}{\phi} = \frac{51}{4}\frac{m^2}{\alpha \ell^2} \text{ ---------- (19)}$$

$$\because \alpha^2 = \left(1 - \frac{6.m^2}{h^2}\right)\alpha = \left(1 - \frac{6.m^2}{h^2}\right)^{\frac{1}{2}}$$

$$\alpha = 1 - \frac{3.m^2}{h^2}$$

$$\alpha = 1 - \frac{3.m^2}{m\ell} \therefore \alpha = 1 - \frac{3.m}{\ell}$$

$$\alpha = 1 - \frac{3x1.47x10^5}{5.544x10^{12}} \quad \alpha = 1 - 0.8x10^{-7} \quad \alpha \approx 1$$

$$\frac{\delta\omega}{\phi} = \frac{51}{4}\frac{m^2}{\alpha \ell^2}$$

$$\delta\omega = \frac{51}{4}\frac{m^2}{\alpha \ell^2}\phi$$

If we consider deviation in α from 1 that is $8x10^{-8}$ then it is seen that for mercury $\delta\omega = (415x360x3600)x8x10^{-8} = 43.024$" which agrees well with the data available.

The calculations of $\frac{6m^2u}{h^2}$ is small as against the $\frac{51}{4}\frac{m^3u^2}{h^2}$. other planets do not give any appreciable precession of the perihelion.

Conclusion:

We find that the first order term is important in the perturbation in calculating the motion of the perihelion .The contribution of the second order term is small as against the . Another important difference is that if the orbit is circular then perturbation vanishes in the anisotropic coordination system and exists only when the orbit is elliptical on the other hand the main contribution in the isotropic coordinate system to perturbation is by the 1st order term and the contribute due to the 2nd order term is negligible. Hence the contribution to the motion of perihelion is negligible in the second order but is mainly due to 1st order in the isotropic system

In the case of mercury the motion of the perihelion is not much different in the isotropic case giving almost the same result as anisotropic and agrees with anisotropic.

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