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Original Research Paper

Fibonacci Sequence and its Special Properties

Mathematics

KEYWORDS: Fibonacci numbers, Fibonacci prime, Fibonacci sequence.

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ABSTRACT

In this paper we will learn about Recurrence Relation. Particularly, Fibonacci sequence and its Properties. Fibonacci numbers and Prime numbers, Fibonacci in plants, Fibonacci numbers in human hand . In conclusion, we will learn three new Properties and its proof using Fibonacci sequence.

Introduction:

When the total number of rabbits for each month is listed, one after the other, it generates the sequence of numbers for which Fibonacci is most famous:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377...

Applications of Fibonacci Numbers

- The Fibonacci number is important in the computational run-time analysis of Euclid's algorithm to determine the greatest common divisor of two integers: the worst case input for this algorithm is a pair of consecutive Fibonacci numbers, Knuth (1997).
- Yuri Matiyasevich was able to show that the Fibonacci numbers can be defined by a Diophantine equation, which led to his original solution of Hilbert's tenth problem.
- The Fibonacci numbers are also an example of a complete sequence. This means that every positive integer can be written as a sum of Fibonacci numbers, where any one number is used once at most.

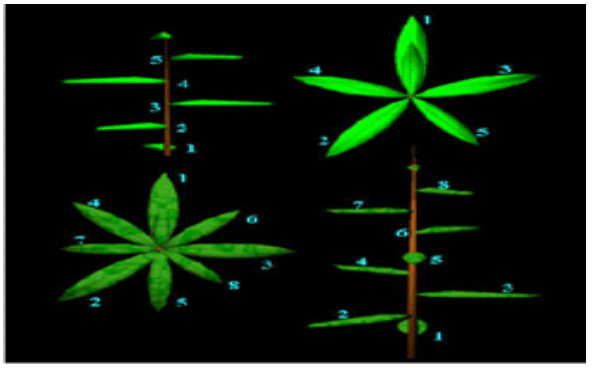
The largest known Fibonacci prime is F_{81839}, which was discovered by David Broadbent and Bouk de Water around 2001 . Each Fibonacci number has been marked in a special way. For instance, if we examine the prime factors of a Fibonacci number, we will see that at least one of the prime factors which have never appeared before as factor in any of the earlier Fibonacci numbers. This is called Carmichael's theorem, in which it is applicable to all Fibonacci numbers F_n.

Fibonacci in Plants

The **Fibonacci** numbers are Nature's numbering system. They appear everywhere in Nature, from the leaf arrangement in plants, to the pattern of the florets of a flower, the bracts of a pinecone, or the scales of a pineapple.

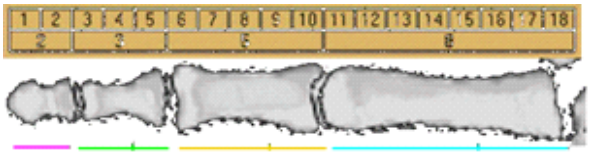
The Leaf Arrangement or Phyllotaxis

Phyllotaxis is the study of the ordered position of leaves on a stem. The leaves on this plant are staggered in a spiral pattern to permit optimum exposure to sunlight. If we apply the Golden Ratio to a circle we can see how it is that this plant exhibits Fibonacci qualities. See on the picture for more detailed illustration of leaf arrangements.[5]



Fibonacci Numbers in Human Hand:

The uniqueness of Fibonacci numbers is that it also occurs in human hand. If we examine our hands very well we would see that each finger is separated into 3 parts with the exception of the thumbs, which is separated into 2 parts. There are 5 fingers on each hand and total of 8 fingers is separated into 3 parts. All these numbers are members of Fibonacci series.



Fibonacci Sequence

Let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, ... Then the sequence can be defined as follows,

$F_{n+1} = F_n + F_{n-1}$, $n \geq 1$.

Property 1:

Let us consider a, b and c are three consecutive Fibonacci numbers then $b^2 - ac = \pm 1$.

Proof.

Case (I)

Take a $= F_3 = F_1 + F_2$

b $= F_4 = F_2 + F_3$

c $= F_5 = F_3 + F_4$

$b^2 - ac = (F_1 + 2F_2)^2 - (F_1 + F_2)(2F_1 + 3F_2)$

$b^2 - ac = -1.$

Case (ii)

Take a $= F_2$

b $= F_3 = F_1 + F_2$

c $= F_4 = F_2 + F_3$

$b^2 - ac = (F_1 + F_2)^2 - F_2(F_1 + 2F_2) = 1$

$b^2 - ac = 1.$

Case (iii) proof by induction on 'n'

$F_n = F_{n-1} + F_{n-2}$

$F_{n+1} = F_n + F_{n-1} = 2F_{n-1} + F_{n-2}$

$F_{n+2} = F_{n+1} + F_n = 3F_{n-1} + 2F_{n-2}$

Take

a $= F_n$, b $= F_{n+1}$, c $= F_{n+2}$.

$b^2 - ac = (F_{n+1})^2 - F_nF_{n+2}$

$= F_{n+1}^2 - F_nF_{n+2}$

$F_{n+1}^2 = (2F_{n-1} + F_{n-2})^2$

$= 4F_{n-1}^2 + F_{n-2}^2 + 4F_{n-1}F_{n-2}$ .....(1

$F_nF_{n+2} = (F_{n-1} + F_{n-2})(3F_{n-1} + 2F_{n-2})$

$= 3F_{n-1}^2 + 5F_{n-1}F_{n-2} + 2F_{n-2}^2$ ...(2

(1) – (2) $\Rightarrow$

$F_{n+1}^2 - F_nF_{n+2} = 4F_{n-1}^2 + F_{n-2}^2 - 4F_{n-1}F_{n-2} - 3F_{n-1}^2 - 5F_{n-1}F_{n-2} - 2F_{n-2}^2$

$= F_{n-1}^2 - F_{n-1}F_{n-2} - F_{n-2}^2$

$= F_n(F_{n-1} - F_{n-2}) - F_{n-1}F_{n-2}$, $n \geq 3$.

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Case (i) For n = odd, suppose n = 3.

$= F_n(F_{n-1} - F_{n-2}) - F_{n-1}F_{n-2} = -1.$

Case (ii) For n = even, suppose n=4.

$= F_n(F_{n-1} - F_{n-2}) - F_{n-1}F_{n-2}$

$= F_4(F_3 - F_2) - F_3F_2 = 1.$

Hence the proof.

Property 2:

Let us consider a, b, c and d are four consecutive Fibonacci numbers then ad – bc = ±1.

Proof.

Case (I)

Take $a = F_1$

$b = F_2$

$c = F_3 = F_1 + F_2$

$d = F_4 = F_2 + F_3$

$= 2F_2 + F_1$

$ad - bc = F_1(2F_2 + F_1) - F_2(F_1 + F_2)$

$ad - bc = 1$

Case (ii)

Take $a = F_2$

$b = F_3 = F_1 + F_2$

$c = F_4 = F_2 + F_3$

$= 2F_2 + F_1$

$d = F_5 = F_3 + F_4$

$= 3F_2 + 2F_1$

$ad - bc = F_2(3F_2 + 2F_1 - (F_1 + F_2)(2F_2 + F_1)$

ad – bc = -1.

Case (iii) proof by induction on 'n'

$F_n = F_{n-1} + F_{n-2}$

$F_{n+1} = F_n + F_{n-1}$

$= 2F_{n-1} + F_{n-2}$

$F_{n+2} = F_{n+1} + F_n$

$= 3F_{n-1} + 2F_{n-2}$

$F_{n+3} = F_{n+1} + F_{n+2}$

$= 5F_{n-1} + 3F_{n-2}$

$a = F_n = F_{n-1} + F_{n-2}$

$b = F_{n+1} = 2F_{n-1} + F_{n-2}$

$c = F_{n+2} = 3F_{n-1} + 2F_{n-2}$

$d = F_{n+3} = 5F_{n-1} + 3F_{n-2}$

$ad - bc = F_nF_{n+3} - F_{n+1}F_{n+2}$

$F_nF_{n+3} = (F_{n-1} + F_{n-2})(5F_{n-1} + 3F_{n-2})$

$= 5F_{n-1}^2 + 8F_{n-1}F_{n-2} + 3F_{n-2}^2$ .....(3)

$F_{n+1}F_{n+2} = (2F_{n-1} + F_{n-2})(3F_{n-1} + 2F_{n-2})$

$= 6F_{n-1}^2 + 7F_{n-1}F_{n-2} + 2F_{n-2}^2$ ...(4)

From (3) and (4)

$ad - bc = 5F_{n-1}^2 + 8F_{n-1}F_{n-2} + 3F_{n-2}^2 - 6F_{n-1}^2 - 7F_{n-1}F_{n-2} - 2F_{n-2}^2$

$= -(F_n(F_{n-1} - F_{n-2}) - F_{n-1}F_{n-2}), n \geq 3.$

Case (i) For n = odd, suppose n = 3

$= -(F_n(F_{n-1} - F_{n-2}) - F_{n-1}F_{n-2})$

$= -(F_3(F_2 - F_1) - F_2F_1)$

$= 1.$

Case (ii) For n = even, suppose n=4.

$= -(F_n(F_{n-1} - F_{n-2}) - F_{n-1}F_{n-2})$

$= -(F_4(F_3 - F_2) - F_3F_2)$

$= -1.$

Hence the proof.

Property 3:

The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$. Then $7F_{n+2}^2 - F_n^2 - F_{n+1}^2$ is divisible by F_{n+3}.

Proof.

$7F_{n+2}^2 - F_n^2 - F_{n+1}^2 =$

$7F_{n+2}^2 + 7F_{n+1}^2 - (F_n^2 - 8F_{n+1}^2)$

$= 7F_{n+3}(F_{n+2}^2 - F_{n+2}F_{n+1} + F_{n+1}^2) - F_{n+3}$

$(F_n^2 - 2F_nF_{n+1} + F_{n+1}^2)$

F_{n+3} is a factor on the right hand side and this completes the proof. We are learned about the basics on Recurrence Relations and Fibonacci sequence. We are learned about Fibonacci sequence and its applications in various fields. Particularly, Rabbit problem, Golden Ratio, Prime numbers, Fibonacci in Plant and Animal.

Mainly, We are learned the following important Properties and theorems related on Fibonacci sequence.

1. The sum of the first n Fibonacci numbers can be expressed as

$$F_1 + F_2 + \ldots + F_{n-1} + F_n = F_{n+2} - 1.$$

2. The sum of the odd terms of the Fibonacci sequence

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$

3. The sum of the even terms of the Fibonacci sequence

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1.$$

4. The sum of the squares of the first n Fibonacci numbers

$$F_1^2 + F_2^2 + F_3^2 + \ldots + F_{n-1}^2 + F_n^2 = F_nF_{n+1}.$$

5. The sum of the Fibonacci numbers with alternating signs

$$F_1 - F_2 + F_3 - F_4 + \ldots + (-1)^{n+1}F_n = (-1)^{n+1}F_{n-1} + 1.$$

6. Difference of Squares of Fibonacci Numbers

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2$$

7. The Fibonacci number F_{sk} is a multiple of 5, for all integers $k \geq 1$.

8. Successive terms of the Fibonacci sequence are relatively prime

9. The sequence of ratio of successive Fibonacci Numbers $\frac{F_{n+1}}{F_n}$ converges to Golden Ratio i.e) $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi.$

10. For the Fibonacci sequence, gcd $(F_n, F_{n+1}) = 1$ for every $n \geq 1$.

11. For $m \geq 1, n \geq 1$, F_{mn} is divisible by F_m

12. (F_m, F_n) If $m = nq + r$, then gcd (F_m, F_n) = gcd (F_r, F_n).

13. The greatest common divisor of two Fibonacci Number is again a Fibonacci Number.

14. F_m is divisible by F_n if and only if m is divisible by n.

15. F_m / F_{mn} for all integers m, n.

Conclusion:

Fibonacci Sequence

Let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, … Then the sequence can be defined as follows, $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$.

Property 1:

Let us consider a, b and c are three consecutive Fibonacci numbers then

$b^2 - ac = \pm 1.$

Property 2 :

Let us consider a, b, c and d are four consecutive Fibonacci numbers then ad – bc = ±1.

Property 3 :

The Fibonacci sequence is defined by

$F_0 = 1,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.
Then $7F_{n+2}^3 - F_n^3 - F_{n+1}^3$ is divisible by F_{n+3}.

These properties may be useful in future and it may be a useful tool in Number theory.

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