Volume : 2 | Issue : 3 | Mar 2013 • ISSN No 2277 - 8179

Research Paper

Mathematics

KEYWORDS : Homotopy perturbation method; Fokker plank equation

Homotopy Perturbation Method for Solving Systems of Linear and Nonlinear Kolmogorov Equations

Aisha Rafi

Department of Mathematics, Faculty of mathematics, Poornima University, Jaipur, India

ABSTRACT

In recent years, many more of the numerical methods were used to solve a wide range of mathematical, physical and engineering problems linear and nonlinear. In this article, we shall use the Homotopy perturbation method (HPM) to solve some systems of linear and nonlinear Fokker-Planck equations and some similar equations by using Homotopy perturbation method are solved. Some examples are solved by Homotopy perturbation method to illustrate the simplicity and reliability of this method.

I. Introduction

The HPM, proposed first by Ji-Huan He [1,2], for solving differential and integral equations, linear and nonlinear, has been the subject of extensive analytical and numerical studies. The aim of this article is to extend the application of the He's HPM to solve linear and nonlinear systems The Fokker–Planck equation. such as the The Fokker–Planck equation describes the time evolution of the probability density function of the velocity of a particle, and can be generalized to other observables as well. It is named after Adriaan Fokker andMax Planck and is also known as the Kolmogorov forward equation (diffusion), named after Andrey Kolmogorov. who first introduced it in a 1931 paper. When applied to particle position distributions, it is better known as the Smoluchowski equation. The case with zero diffusion is known in statistical mechanics as Liouville equation In this paper, we apply the Homotopy perturbation method (HPM) for solve linear and nonlinear FPEs. The general form of Fokker-Planck equation (FPE) for variables X and t is as follows [6]:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]u(x,t), \tag{1}$$

With the following initial condition:

$$u(x,0) = f(x), \qquad x \in R.$$

Here $B(x) > 0$ is called the diffusion coefficient and $A(x) > 0$ the drift coefficient. The diffusion and drift coefficients can also be functions of x and t, i.e.

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]u(x,t), \tag{2}$$

s Eq. (1) is an equation for the motion of the concentration field $u(x,t)$. Mathematically, this equation is a linear second-order partial differential equation of parabolic type. Eq. (1) is also called forward Kolmogorov equation. The backward Kolmogorov equation is written in the following form:

$$\frac{\partial u}{\partial t} = -\left[A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2}\right]u(x,t), \tag{3}$$

A generalized form of Eq. (1) to N variables $x_1, x_2, \dots, x_N$ can be written as follows:

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x)\right]u(x,t), \tag{4}$$

With the following initial condition

$u(x,0) = f(x), \qquad x = (x_1, x_2, \dots, x_N) \in R^N$. Generally in Eq.(4) drift vector A_i and diffusion tensor $B_{i,j}$ depend on N variables $x_1, x_2, \dots, x_N$.

The nonlinear FPE for one variable is in the following form:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]u(x,t), \tag{5}$$

Eq. (5) for N variables $x_1, x_2, \dots, x_N$ is in the following form:

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x,t,u) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x,t,u)\right]u(x,t), \qquad x = (x_1, x_2, \dots, x_N) \in R^N. \tag{6}$$

The paper is organized as follows: in the next section, the Homotopy perturbation method is introduced. Homotopy perturbation method

We consider the following nonlinear differential equation:

L(u) + N(u) = f(r), r ∈ Ω, (7)

With the boundary $B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma,$

Where L is a linear operator, N is a nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and f(r) is a known analytic function.

By the Homotopy perturbation technique [1,2], He construct a homotopy:

v(r, p) : Ω× [0, 1] → R which satisfies:

H(v, p) = (1 − p) [L(v) − L(u0)] + p [L(v) + N(v) − f(r)] = 0, (8)

H(v, p) = L(v) − L(u0) + pL(u0) + p [N(v) − f(r)] = 0, (9)

where r ∈ Ω, p ∈ [0, 1] is an impending parameter and u0 is an initial approximation

which satisfies the boundary conditions. Obviously, from Eqs. (8) and (9), we have:

The changing process of p from zero to unity is just of v(r, p) from u0(r) to u(r). In topology, this is called deformation, L(v) − L(u0) and L(v) + N(v) − f(r) are called homotopic. The basic assumption is that the solution of Eqs.(8) and (9) can be expressed as a power series in p :

v = v0 + pv1 + .p2v2+ .. .

The approximate solution of Eq. (7), therefore, can be readily obtained:

$$u = \lim_{p\to 1} v = v0 + pv1 + .p2v2+ . \tag{10}$$

The convergence of the series (10) has been proved in [1,2].

Consider the following nonlinear partial differential equation with time derivative of any order [8]:

$$D_t^n u(x,t) = L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + f(x,t), \quad t > 0,$$

Where L is a linear operator, N is a nonlinear operator and f is a known analytic function,

Subject to the initial conditions:

$$\frac{\partial^m}{\partial t^m}u(x,0) = h_m(x), \quad m = 0, 1, 2, ..., n-1. \tag{11B}$$

In view of the Homotopy technique, we can construct the following Homotopy:

$$\frac{\partial^n u}{\partial t^n} - L(u, u_x, u_{xx}) - f(x,t) = p\left[\frac{\partial^n u}{\partial t^n} + N(u, u_x, u_{xx}) - D_t^n u\right]. \tag{11C}$$

$$\frac{\partial^n u}{\partial t^n} - f(x,t) = p\left[\frac{\partial^n u}{\partial t^n} + L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) - D_t^n u\right],$$

Volume : 2 | Issue : 3 | Mar 2013 • ISSN No 2277 - 8179

Where p ∈ [0, 1]. The Homotopy parameter p always changes from zero to unity. When p = 0, Eq. (11C) & (11D) becomes

$$\frac{\partial^n u}{\partial t^n} = L(u, u_x, u_{xx}) + f(x,t),$$

Linearized equation:

$$\frac{\partial^n u}{\partial t^n} = f(x,t),$$

p = 1, Eq. (11C) or Eq. (11D) turns out to be the original differential Eq. (11A). The basic assumption is that the solution of Eq. (11C) or Eq. (11D) can be written as a power series in p

$$u = u_0 + pu_1 + p^2 u_2 + \dots . \tag{13}$$

Finally, we approximate the solution u(x, t) by:

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t).$$

For illustrating the HPM in solving systems of, linear and nonlinear Fokker-Planck equations and some similar equations by using Homotopy perturbation method are solved. We consider the following system.

2. APPLICATION

Suppose that X and Y be Banach space and $N: X \to Y$ is a contraction nonlinear mapping, that is

$$\forall\, v, \bar{v} \in X: \quad \|N(v) - N(\bar{v})\| \le \gamma \|v - \bar{v}\| \qquad 0 < \gamma < 1$$

Which according to Banach's fixed point theorem, having the fixed point u, that is. $n_{u=(u)}$

The sequence generated by the Homotopy perturbation method will be regarded as

$$V_n = N(V_{n-1}), \qquad V_{n-1} = \sum_{i=0}^{n-1} u_i, \qquad n = 1,2,3 \dots .$$

and suppose that $V_0 = v_0 = u_0 \in B_r(u)$ where $B_r(u) = \left\{u^* \in \frac{X}{\|u^* - u\|} < r\right\}$, then we have the following statements:

(i) $\|V_n - u\| \le \gamma^n \|v_0 - u\|$.

(ii) $V_n \in B_r(u)$.

(iii) $\lim_{n\to\infty} V_n = u$.

Proof: (i) By the induction method on n, for $n = 1$ we have

$$\|V_1 - u\| = \|N(V_0) - N(u)\| \le \gamma \|v_0 - u\|.$$

Assume that $\|V_{n-1} - u\| \le \gamma^{n-1} \|v_0 - u\|$ as an induction hypothesis, then

$$\|V_n - u\| = \|N(V_{n-1}) - N(u)\| \le \gamma \|V_{n-1} - u\| \le \gamma\gamma^{n-1}\|v_0 - u\| = \gamma^n \|v_0 - u\|.$$

(ii) Using (i), we have

$$\|V_n - u\| \le \gamma^n \|v_0 - u\| \le \gamma^n r < r \Rightarrow V_n \in B_r(u).$$

(iii) Because of $\|V_n - u\| \le \gamma^n \|v_0 - u\|$, and, we drive $\lim_{n\to\infty} \|V_n - u\| = 0$, that is $\lim_{n\to\infty} V_n = u$.

1- At first, we construct a homotopy perturbation method for equation (1) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]v\right) = 0 \tag{15}$$

$$\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u_0}{\partial t} - \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]v\right) = 0 \tag{16}$$

By substituting (13) into (16) and equating the coefficients of like terms with the identical powers of p, we obtain:

$$p^0: \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t} \tag{17}$$

$$p^1: \frac{\partial v_1}{\partial t} = -\frac{\partial u_0}{\partial t} + \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]v_0$$

$$= -\frac{\partial u_0}{\partial t} + \left(-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2}\right)v_0 + \left(-A(x) + 2\frac{\partial B(x)}{\partial x}\right)\frac{\partial v_0}{\partial x} + B(x)\frac{\partial^2 v_0}{\partial x^2} \tag{18}$$

$$p^2: \frac{\partial v_2}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]v_1$$

$$= \left(-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2}\right)v_1 + \left(-A(x) + 2\frac{\partial B(x)}{\partial x}\right)\frac{\partial v_1}{\partial x} + B(x)\frac{\partial^2 v_1}{\partial x^2} \tag{19}$$

$$p^k: \frac{\partial v_k}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]v_{k-1}$$

$$= \left(-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2}\right)v_{k-1} + \left(-A(x) + 2\frac{\partial B(x)}{\partial x}\right)\frac{\partial v_{k-1}}{\partial x} + B(x)\frac{\partial^2 v_{k-1}}{\partial x^2} \tag{20}$$

By integration of the both sides of the equations, we obtain the following multiple solutions:

$$v_0 = u_0 \tag{21}$$

$$v_k = \int_0^t \left[\left(-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2}\right)v_{k-1} + \left(-A(x) + 2\frac{\partial B(x)}{\partial x}\right)\frac{\partial v_{k-1}}{\partial x} + B(x)\frac{\partial^2 v_{k-1}}{\partial x^2}\right]dt$$

$$k = 1,2,3, \dots \tag{22}$$

2- We construct a homotopy perturbation method for equation (2) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]v\right) = 0 \tag{23}$$

$$\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u_0}{\partial t} - \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]v\right) = 0 \tag{24}$$

By substituting (13) into (24) and equating the coefficients of like terms with the identical powers of p p, we obtain:

$$p^0: \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t} \tag{25}$$

$$p^1: \frac{\partial v_1}{\partial t} = -\frac{\partial u_0}{\partial t} + \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]v_0$$

$$= \left(-\frac{\partial A(x,t)}{\partial x} + \frac{\partial^2 B(x,t)}{\partial x^2}\right)v_0 + \left(-A(x,t) + 2\frac{\partial B(x,t)}{\partial x}\right)\frac{\partial v_0}{\partial x} + B(x,t)\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial u_0}{\partial t}$$

$$p^2: \frac{\partial v_2}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]v_1 \tag{26}$$

$$= \left(-\frac{\partial A(x,t)}{\partial x} + \frac{\partial^2 B(x,t)}{\partial x^2}\right)v_1 + \left(-A(x,t) + 2\frac{\partial B(x,t)}{\partial x}\right)\frac{\partial v_1}{\partial x} + B(x,t)\frac{\partial^2 v_1}{\partial x^2} \tag{27}$$

$$p^k: \frac{\partial v_k}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]v_{k-1}$$

$$= \left(-\frac{\partial A(x,t)}{\partial x} + \frac{\partial^2 B(x,t)}{\partial x^2}\right)v_{k-1} + \left(-A(x,t) + 2\frac{\partial B(x,t)}{\partial x}\right)\frac{\partial v_{k-1}}{\partial x} + B(x,t)\frac{\partial^2 v_{k-1}}{\partial x^2}$$

By integration of the both sides of the equations, we obtain the following multiple solutions:

$$v_0 = u_0 \tag{29}$$

$$v_k = \int_0^t \left[\left(-\frac{\partial A(x,t)}{\partial x} + \frac{\partial^2 B(x,t)}{\partial x^2}\right)v_{k-1} + \left(-A(x,t) + 2\frac{\partial B(x,t)}{\partial x}\right)\frac{\partial v_{k-1}}{\partial x} + B(x,t)\frac{\partial^2 v_{k-1}}{\partial x^2}\right]dt$$

$$k = 1,2,3, \dots \tag{30}$$

Conclusion:

We solved the Fokker-Planck equation by homotopy perturbation method. We notice that the HPM is very accurate method since the results of this method are the same results of ADM and VIM. So that the HPM is remarkably effective for solve the Fokker-Planck equation

REFERENCE

[1] J.H. He, Comput. Methods Appl. Mech. Engrg. 178 (1999) 257. | [2] J.H. He, Int. J. Non-Linear Mech. 35(1) (2000) 37. | [3] A. R. Abd-Ellateef Kamar, Monotone iteration technique for singular perturbation problem, Appl. Math. Comp. 131 (2002) 559-571. | [4] F. Geng, M.Cui., B. Zhang, Method for solving nonlinear initial value problems by combining Homotopy perturbation and reproducing kernel Hilbert space methods, Nonlinear Analysis: Real World Applications 11 (2010) 637-644. | [5] F. Abidi, K. Omrani, The homotopy analysis method for solving the Fornberg-Whitham equation and comparison with adomian's decomposition method, Comput. Math. Appl. 59(2010) 2743-275. | [6] J. Biazar, P. Gholamin and K. Hosseini, Variational iteration method for solving Fokker-Planck equation, Journal of the Franklin Institute 347 (2010) 1137-1147. | [7] S. Abbasbandy, Appl. Math. Comput. 173 (2006) 493. | [8] S. Momani, Z. odibat, Phys. Lett. A 365 (2007) 345. |