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The Study of Common Fixed Points in Complex Valued Symmetric Spaces

Mathematics

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ABSTRACT

We introduce complex valued symmetric spaces and obtain sufficient conditions for the existence of common fixed points of a pair of weakly compatible mappings. Our results extend the results of Pant.

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1. INTRODUCTION

Presence of fixed points is an intrinsic property of a self-map. There are many conditions for the existence of such points which involve a mixture of algebraic, order theoretic and topological properties of mappings or its domain. Since the appearance of the Banach contraction mapping principle, a number of articles have been dedicated to the improvement and generalization of that result. Most of these deal with the generalizations of the contractive condition in metric spaces. Ghaler [5] generalized the idea of metric space and introduced a 2-metric space which was followed by a number of papers dealing with this generalized space. Plenty of material is also available in other generalized metric spaces, such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, Quasi metric spaces, Quasi semi metric spaces, D-metric spaces, cone metric spaces, Complex valued metric space. (see [2, 3, 4, 5, 6, 7, 8, 11, 12, 18, 19]).

In this paper, we extend some common fixed point theorems for weakly compatible and non-compatible maps on complex valued symmetric space. Our results extend the results of Azam et al. [2], Klin-eam et al. [10], Sintunavarat et al.[17] and Pant [14, 15] .

We begin with the following basic definitions of concepts relating to complex valued symmetric spaces for ready reference and also for the sake of completeness.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\leq$ on $\mathbb{C}$ as follows:

$z_1 \leq z_2$ if and only if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$; $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$: It follows that $z_1 \leq z_2$: If one of the following conditions is satisfied:

(i). $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$; $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$; (ii). $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$; $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;

(iii). $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$; $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$;(iv). $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$; $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

Definition 1.1.[2] : Let X be a non-empty set. Suppose that the mapping $d: X \times X \to \mathbb{C}$, satisfies:

i. $0 \leq d(x, y)$, $\forall x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;

ii. $d(x, y) = d(y, x)$ for all $x, y \in X$;

iii. $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Now we are defining complex valued symmetric space (i.e. without using triangle inequality). We are showing by example that the complex valued symmetric space is more general than complex valued metric space [2].

Definition 1.1.[2] : Let X be a non-empty set. Suppose that the mapping $d: X \times X \to \mathbb{C}$, satisfies;

i. $0 \leq d(x,y), \forall x,y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$;

ii. $d(x,y) = d(y,x)$ for all $x, y \in X$;

Example: Let $X = \mathbb{C}$ with the complex valued symmetric

$$d(z_1, z_2) = i|(z_1 - z_2)^2|.$$

Where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in X$.

Remark 1.1. Note that in the above example the function d is not a complex metric. In this example the triangle inequality property of complex metric will not be satisfied. Therefore in this example the space (X, d) is not a complex valued metric space. Here the space is complex valued symmetric space. Hence complex valued symmetric space is more general than complex valued metric space.

Definition 1.3 [20]. Let (X, d) be a complex valued symmetric

W_1: Given $\{x_n\}$, x and y in X, $d(x_n, x) \to 0$ and $d(x_n, y) \to 0 \Rightarrow x = y$.

W_2: Given $\{x_n\}$, $\{y_n\}$ and an x in X, $d(x_n, x) \to 0$ and $d(x_n, y_n) \to 0 \Rightarrow d(y_n, x) \to 0$.

Definition 1.4 [9]. Two self- maps S and T are said to be weakly compatible if they commute at the coincidence points. i.e. if Tu = Su for some u $\in$ X, then TSu = STu.

Definition 1.4 [1]. A pair of self-mappings S and T on a complex valued symmetric space (X, d) is said to enjoy property $(E.A.)$ if there exists a sequence $\{x_n\}$ in X such that such that for some $t \in X$. $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

2. MAIN RESULT

Let (X, d) be a complex valued symmetric *space* which satisfies W_1 condition. Let S and T be a self mapping of a symmetric space (X, d) and let $\overline{SX}$ denote the closure of the range of S. We now prove the following theorem.

Theorem-2.1. Let S and T are weakly compatible self mappings of a complex valued symmetric space (X, d) and S and T satisfy property $(E.A.)$ such that,

$$\overline{SX} \subset TX \qquad (1)$$

$$d(Sx, Sy) \le kd(Tx, Ty), k \ge 0 \quad (2)$$

$$d(x, Tx) < \max\{d(x, Sx), d(Tx, Sx)\}, \quad (3)$$

whenever the right-hand side is non-zero. Then S and T have a common fixed point. Moreover, all *fixed points* of S are *fixed points* of T.

Proof: Since S and T are satisfies property $(E.A.)$, there exists a sequence $\{x_n\}$ such that $Sx_n \to t$ and $Tx_n \to t$ for some t in X. Then since $t \in \overline{SX}$ and $\overline{SX} \subset TX$ there exists x_0 in X such that $t = Tx_0$. By (2) we get $d(Sx_n, Sx_0) \le kd(Tx_n, Tx_0)$, On letting $n \to \infty$, $\lim_{n\to\infty} d(Sx_n, Sx_0) = 0$. Now Since $Sx_n \to t = Tx_0$ and $Sx_n \to Sx_0$. Hence by the condition W_1 we get $Sx_0 = Tx_0$. It follows that, weak compatibility of S and T implies that $STx_0 = TSx_0$. Also $SSx_0 = STx_0 = TSx_0 = TTx_0$. We claim that $TSx_0 = Sx_0$. If not by virtue of (3) we get

$$d(Sx_0, TSx_0) < \max\{d(Sx_0, SSx_0), d(TSx_0, SSx_0)\}$$

$$= d(Sx_0, SSx_0) = d(Sx_0, TSx_0).$$

This is a contradiction. Hence $Sx_0 = SSx_0 = TSx_0$ and Sx_0 is a *common fixed point* of S and T. We claim that every *fixed point* of S is a *fixed point* of T. Let $Sy_0 = y_0$ Suppose $Ty_0 \ne y_0$, Then since $Max\{d(y_0, Sy_0)\} = d(y_0, Ty_0) > 0$.

Hence by condition (3) we get,

$$d(y_0, Ty_0) < Max\{d(y_0, Sy_0), d(Ty_0, Sy_0)\}$$

$$< Max\{d(y_0, y_0), d(Ty_0, y_0)\}$$

$$< d(Ty_0, y_0).$$

So $Ty_0 = y_0$. This is a contradiction. Hence every *fixed point* of S is also a *fixed point* of T.

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