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Research Paper

Total Zero Divisor Graph of a Commutative Ring

Mathematics

KEYWORDS : Zerodivisors, Total zerodivisor graph, commutative ring, connected graph

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ABSTRACT

Let R be a commutative ring with Z(R), its set of zero divisors. The total zero divisor graph of R, denoted Z(Γ(R)) is the undirected (simple) graph with vertices Z(R)=Z(R)-{0}, the set of nonzero zero divisors of R. and for distinct x, y ∈ z(R)*, the vertices x and y are adjacent if and only if x + y ∈ z(R). In this paper, we study if Z(Γ(R)) is finite and every vertex of Z(Γ(R)) has a finite degree then R is finite and also prove that Z(Γ(R)) connected with diam ≤ 3.*

1. Introduction:

Let R be a commutative Ring with non zero unity. The concept of the graph of the zero divisors of R was first introduced by Beck [1], where he was mainly interested in coloring. In his work all elements of the ring were vertices of the graph. The investigation of colorings of a commutative ring was then continued by D. D. Anderson and Naseer [2], In [3], D. F. Anderson and Livingston associate a graph, Γ(R), to R with vertices $Z(R)^*=Z(R)\backslash\{0\}$, the set of non zero zero divisors of R, and for distinct x,y $Z(R)\backslash\{0\}$. The vertices x and y are adjacent if xy=0. In [5] D.F. Anderson and Badawi introduced the total graph of R, denoted by T(Γ(R)) as the graph with all elements of R as vertices, and for distinct x, y R are adjacent if x+y Z(R), they studied some graphical parameters of this graph such as diameter and girth.

We studied the total zero divisor graph is the (undirected) graph with vertices $Z(R)^*=Z(R)-\{0\}$. The set of non zero zero divisors of R and for distinct x,y $Z(R)^*$, the vertices x and y are adjacent if and only if x+y Z(R). It is denoted by Z(Γ(R)) and is the (induced) sub graph of total graph. In this paper we show that Z(Γ(R)) is finite then R is finite and not an integral domain, if every vertex of Z(Γ(R)) has finite degree then R is finite and also prove that Z(Γ(R)) is connected with diam ≤ 3.For some other recent papers on zero divisor graphs,see[7,8,9-11,13,14]. for algebraic preliminaries and definitions see[12].for develop this paper we use some results of [6,15].

Preliminaries:

Degree: The number of edges incident with a vertex v is called the degree of v and it is denoted by d(v). the minimum and maximum degrees of a vertex of a graph are respectively denoted by δ and Δ.

Complete Graph: A graph G in which every vertex is adjacent to every other vertex is called a complete graph. Complete graph is represented as K_n where n is the number of vertices in K_n .

Connected Graph: A graph G is said to be a connected graph if there is at least one path between every pair of vertices in G. otherwise G is said to be a disconnected graph.

Distance: Any two distinct vertices a and b in graph G , the distance between a and b, denoted by d(a,b) is the length of a shortest path connecting a and b ,if such a path exist. Otherwise d(G) =

Diameter of G : diam(G) = Sup{d(x,y) x & y are distinct vertices in G},where d(x,y) is the length of shortest path from x to y in G. if there is no such a path then d(x,y) = .

The girth of G: The girth of G is denoted by gr(G) is length of shortest cycle in G. if G contains no cycles the gr(G) =

Path: A trail in which all the vertices are distinct is called a path.

Cycle: A path whose origin and terminus vertices are the same is called a cycle.

Main Results:

Theorem:1

Let R be a commutative ring then the total zero divisor graph is finite iff either R is finite or an integral domain. In particular if $1 \le Z(\Gamma(R)) < \infty$.Then R is a finite and not a field.

Proof:

Let R be a commutative ring and Z(R) be the set of zero divisors in R.

Let Z(Γ(R)) be Total zero divisor graph. then all vertices of Z(Γ(R)) is non zero zero divisors of R.

It is trivial that if R is finite then Z(Γ(R))is also finite.

Suppose that Z(Γ(R))is finite and non empty. This implies that Z(R) is finite.

Suppose these are two elements x, y R, $x \neq 0$, $y \neq 0$. Such that x+y Z(R) for some z R s.t. (x+y)z=0.

Let I=Ann(z), then x+y I.

Since x+y Z(R) This implies that $I \subseteq Z(R)$ further I is finite and $r(x+y) \in I \ \forall \ r \in R$

$[\because x+y \in I, r \in R \Rightarrow r(x+y) \in I]$ Suppose R is finite.

Then there is an i I such that $J = \{r \in R / r(x+y) = i\}$ is infinite.

For any r,s J, r(x+y) = i

$$s(x+y) = i$$

$$(r-s)(x+y) = 0$$

$$(r-s) \in \mathrm{Ann}(x+y)$$

$$[\ \because\ J \subset \mathrm{Ann}(x+y)\ ,\ J \text{ is infinite}\ \Rightarrow r-s \in J \Rightarrow r-s \in \mathrm{Ann}(x+y)\ ,\ J \subset \mathrm{Ann}(x+y)\ ]$$

Since $r-s \in Z(R)$

i.e. $Ann(x+y) \subseteq Z(R)$, is infinite, a contradiction therefore R must be finite. ♦

Theorem:2

If R is a ring such that R is not an integral domain and every vertex of Z(Γ(R)) has a finite degree then R is a finite ring.

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Proof:-
Let R is a ring such that R is not an I.D and every vertex of $Z(\Gamma(R))$ has finite degree.

We show that R is a finite ring

Let us assume that R is an infinite ring.

Let x and y be non zero elements of R s.t. x+y Z(R).

Then for some z Z(R) s. t. (x+y)z=0

Then $zR^* \subseteq Ann(x+y)$

If zR^* is infinite then x+y has infinite degree in Z((R)) , a contradiction since every vertex of Z((R)) is finite.

If zR^* is finite then there exist an infinite subset A of R^* such that a_1, a_2 A then $a_1z = a_2z$.

If a_0 is a fixed element of A then$\{a_0 - aaA\}$is an infinite subset of Ann(z)

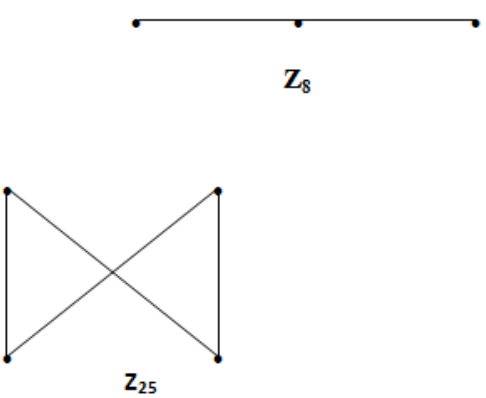
$[\ a_0, a_1Aza_0 = za_1\ z(a_0-a_1)=0 \quad a_0-a_1A\ ]$

And so z has infinite degree in Z((R)),which is a contradiction to every vertex of Z((R)) is finite.

Thus R is finite Ring. ♦

Remark: if R is not an integral Domain then we have $|Z((R))| < |R| <$ every vertex of Z((R)) has finite degree.

Example :



Lemma:1 If axb is a path in Z((R)) then either {0,x} is an ideal in R or axb is connected in a cycle of length ≤ 4.

Proof: Let ann(a) = { yRya =0}

Either ann(a)ann(b) ={0,x} or there is cx,0 with a+c Z(R) ,b+c Z(R).

In the first case {0,x} is an ideal,

In the second case axbca is a cycle of length ≤ 4. ♦

Theorem :3 Let R be commutative ring such that Z(R) is not an ideal of R then Z((R)) is connected with diam Z((R)) = 2.

Proof: Each x Z(R)* is adjacent to 0. Thus x0y is a path in Z((R)) of length two between any two distinct x,y Z(R)* .

Moreover, there are non adjacent x,y Z(R)* since Z(R) is not an ideal of R .

So diam Z((R)) = 2. ♦

Next we prove that Z((R)) is connected with diam ≤ 3.

Theorem 4: R be a commutative ring then Z((R)) is connected with diam ≤ 3.

Proof : Let x,y be vertices in Z((R)),

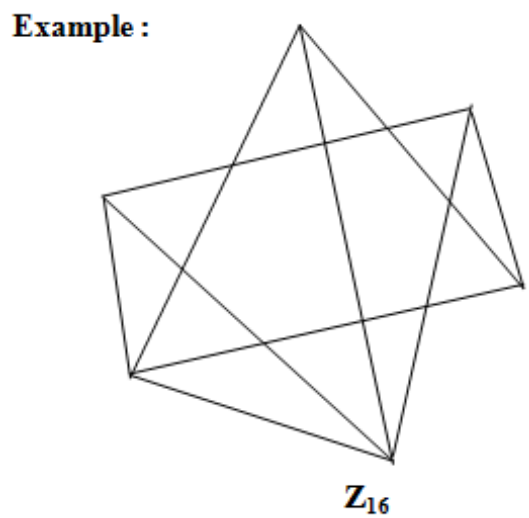
There exists z 0,w0 with x+z Z(R),y+w Z(R).

If x+y Z(R) then xy is a path of length 1 containing x,y.

If x+y Z(R) and w+z Z(R) then x and y are connected by a path xwy of length ≤ 3.

If x+y Z(R) and w+z Z(R) then x and y are connected by a path xy. of length = 2. ♦

Example :



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