Volume : 3 | Issue : 4 | April 2014 • ISSN No 2277 - 8160

Research Paper | Engineering

Elliptic Functions With A View Toward Jacobi Hyperbolic Functions

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ABSTRACT

In Complex Analysis an elliptic function is a doubly periodic meromorphic function which is determined by its value on a fundamental parallelogram which then repeat in a lattice. Elliptic functions must have at least two poles in a fundamental parallelogram.

Jacobi elliptic functions can be expressed in various forms namely as a Trigonometric functions, Theta functions, as a solution of first and second order differential equations etc. In the present paper a new concept of Jacobi Hyperbolic functions using Hyperbola is being developed. Jacobi Hyperbolic functions are related to elliptic functions with complex arguments, just as hyperbolic sines and cosines functions are related to sines and cosines of complex argument. Just like Jacobi elliptic functions, Jacobi Hyperbolic functions are meromorphic functions that are periodic in two directions and can also be represented in various other functions.

In the present study Jacobi Hyperbolic functions are being presented in the form of Trigonometric functions, which are also used as the solutions of first and second order differential equations.

KEYWORDS : Elliptic function, Jacobi Elliptic function, Jacobi Hyperbolic function, Doubly periodic.

1. Introduction

Historically, the starting point of theory of elliptic functions were the elliptic integral of the form (∫ R(t), √P(t))dt, where R is a rational function and P is a polynomial of degree 3 or 4 without multiple factors. The idea of inverting elliptic integrals to obtain elliptic functions is due to Gauss, Abel and Jacobi. The real innovation came around 1800, when Gauss realized that one should not study the elliptic intergrals. The theory of elliptic intergrals was thus derived as a byproduct of the theory of elliptic functions.

Elliptic functions were first discovered by Niels Henrick and their theory was improved by Carl Gustav Jakob Jacobi in 1829. N.H Abel affirms that an elliptic function has meromorphic continuation into the entire complex plane. In addition to an obvious real period, Abel discovered a hidden complex period. So elliptic functions turned out to be doubly periodic functions. This doubly periodicity of elliptic functions was discovered by Gauss which is one of the key property of elliptic functions. A meromorphic function in the plane with two independent periods is also called elliptic. The most important contribution to the theory of elliptic functions after Abel and Jacobi were doubtlessly made by K.Weistrass β (1815 – 1897). Theoretically simpler construction of the field of elliptic functions in which one takes as generators the function ℙ and its derivative was made by K.Weistrass in the 1870s. He replaced Jacobi elliptic functions by the single function which in many respect behaves in a simple way. Likewise, he replaced the Jacobi Theta functions by other functions called Sigma functions, which exhibits analogous advantages. But Jacobi elliptic functions are used more often then Weistrass functions. Jacobi basis the theory of elliptic functions on the inverse functions.

In the present paper two terminologies are being presented.

2.Existing terminology

Existing terminology related to Jacobi functions is discussed below:

2.1.Jacobi elliptic functions

In complex analysis, the Jacobi elliptic functions are a set of basic elliptic functions that have useful analogies to the functions of trigonometry, as indicated by the matching notation sn for sin

Elliptic functions are functions of two variables. The first variable might be given in terms of the amplitude φ, or more commonly, in terms of u . . The second variable might be given in terms of the parameter m, or as the elliptic modulus k, where $k^2 = m$, or in terms of the modular angle α, where $m = \sin^2 \alpha$. There are twelve Jacobian elliptic functions. Each of the twelve corresponds to an arrow drawn from one corner of a rectangle to another. The three basic functions are denoted by cn(u, k), dn(u, k), and sn(u, k) where k is known as the elliptic modulus. They arise from the inversion of the elliptic integral of the first kind,

$$u = F(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}},$$

Where $0 < k^2 < 1$ and k = mod u which is called elliptic modulus, and φ=am (u, k) =am (u) which is the Jacobi amplitude, given by

$\varphi = F^{-1}(u, k) = \text{am}(u, k)$
From this, it follows that
$\sin \varphi = \sin(\text{am}(u, k)) = \text{sn}(u, k)$
$\cos \varphi = \cos(\text{am}(u, k)) = \text{cn}(u, k)$
These functions are doubly periodic generalizations of the trigonometric functions satisfying
sn(u, 0) = sin u
cn(u, 0) = cos u
dn(u, 0) = 1

The other **9** Jacobi elliptic functions are given as below:
ns(u) = 1/sn(u), nc(u) = 1/cn(u),
nd(u) = 1/dn(u), sc(u) = sn(u)/cn(u),
sd(u) = sn(u)/dn(u), dc(u) = dn(u)/cn(u),
ds(u) = dn(u)/sn(u), cs(u) = cn(u)/sn(u),
cd(u) = cn(u)/dn(u).

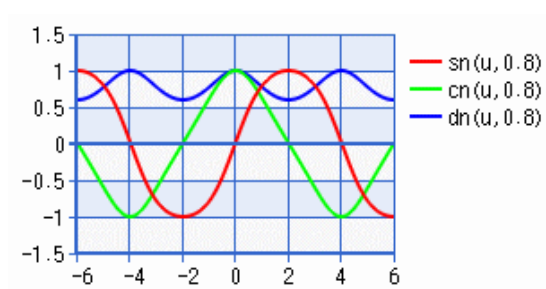


Fig.1. A Graph of the three basic Jacobi elliptic functions sn(u,k), cn(u,k) and dn(u,k)

2.2.Addition Theorems

The function satisfy the
$\text{cn}^2(u, k) + \text{sn}^2(u, k) = 1$
$\text{dn}^2(u, k) + k^2 \text{sn}^2(u, k) = 1$

2.3.Jacobi elliptic functions as a solution of Non-Linear ordinary differential equations

The derivatives of the three basic Jacobi elliptic functions are:
d/dz sn(z)=cn(z) dn(z)
d/dz cn(z)=-sn(z) dn(z)
d/dz dn(z)=-k2 sn(z) cn(z)

With the addition theorems above and for a given k with 0 < k < 1, Jacobi elliptic functions therefore are solutions to the following nonlinear ordinary differential equations:

(i) sn (x) solves the differential equations
a)

$$\frac{d^2y}{dx^2} + (1+k^2)y - 2k^2 y^3 = 0$$

b)

$$\left(\frac{dy}{dx}\right)^2 = (1-y^2)(1-k^2y^2)$$

(ii) cn (x) solves the differential equations
a)

$$\frac{d^2y}{dx^2} + (1-2k^2)y + 2k^2y^3 = 0$$

b)

$$\left(\frac{dy}{dx}\right)^2 = (1-y^2)(1-k^2+k^2y^2$$

(iii) dn (x) solves the differential equations
a)

$$\frac{d^2y}{dx^2} - (2-k^2)y + 2y^3 = 0$$

b)

$$\left(\frac{dy}{dx}\right)^2 = (y^2-1)(1-k^2-y^2)$$

3. Proposed terminology
In the present study Jacobi Hyperbolic functions are being presented in the form of Trigonometric functions

3.1.Jacobi Hyperbolic Functions. Jacob hyperbolic functions are doubly periodic meromorphic functions which are analogous to Jacobi elliptic functions. Jacobi hyperbolic functions are defined in a similar manner as Jacobi elliptic functions. But representation of Jacobi hyperbolic functions in terms of Trigonometric functions is being presented in this paper.

3.2.Trigonometry of Hyperbola
Hyperbola equation is:-

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Normalizing the above equation by choosing b = 1, we get

$$\frac{x^2}{a^2} - y^2 = 1 \qquad (1)$$

Now the three basic hyperbolic functions are defined as sh(u, k), ch(u, k) and dh(u, k). Where u is the argument of hyperbolic functions and k is the modulus of corresponding hyperbola given by

$$u = \oint_p^q r d\theta$$

$$k = e = \frac{\sqrt{a^2+1}}{a}$$

With the argument and modulus of hyperbolic functions defined, the functions themselves are just ratios, as in case of trigonometry

shu = sh(u, k) = y
chu = ch(u, k) = x/a
dhu = dh(u, k) = r/a

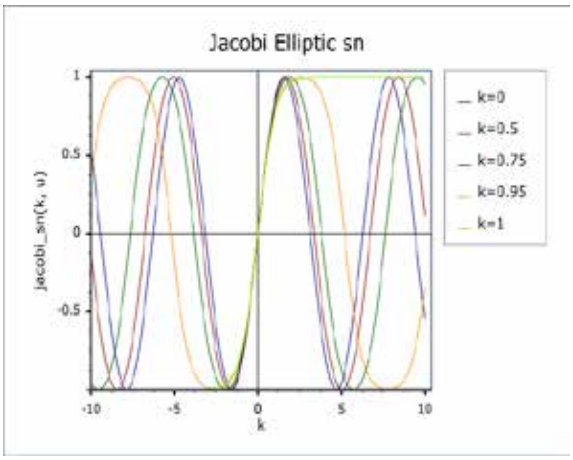


Fig.2. A Graph showing relation between sn and sh (for k = 1).

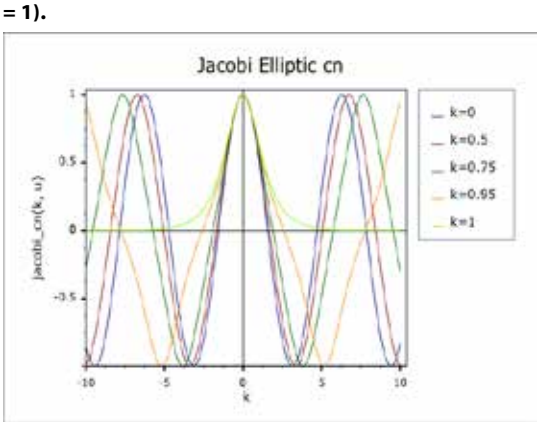


Fig.3. A Graph showing relation between cn and ch (for k = 1).

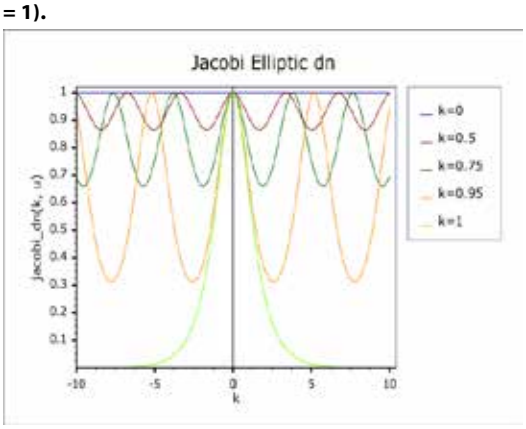


Fig.4. A Graph showing relation between dn and dh (for k = 1).

The other 9 Jacobi hyperbolic functions are given by
hs(u) = 1/sh(u), hc(u) = 1/ch(u),
hd(u) = 1/dh(u), sc(u) = sh(u)/ch(u),
sd(u) = sh(u)/dh(u), dc(u) = dh(u)/ch(u),
ds(u) = dh(u)/sh(u), cs(u) = ch(u)/sh(u),
cd(u) = ch(u)/dh(u).

3.3.Subtraction Theorems
The functions satisfies the two algebraic relations

$ch^2(u, k) - sh^2(u, k) = 1$
$dh^2(u, k) - k^2 sh^2(u, k) = 1$

3.4.Jacobi hyperbolic functions as a solution of Non-Linear ordinary differential equations

We know that,

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$

$$d\theta = \frac{1}{r^2}(x\,dy - y\,dx)$$

But du = rdθ =
From eq (1)

$$\frac{x dx}{a^2} - y dy = 0$$

$$dy = \frac{x}{a^2 y}dx$$

or

$$dx = \frac{a^2 y}{x}dy$$

The corresponding substitution for du is:

$$du = \frac{1}{r}\left[\frac{x2}{a^2 y} - y\right]dx$$

or

$$du = \frac{1}{r}\left[x - \frac{a2y2}{x}\right]dy$$

With these substitutions we get the formulas for differentiating hyperbolic functions.

$$\frac{\mathrm{d}}{\mathrm{du}}\mathrm{chu} = \mathrm{shu}\ \mathrm{dhu}$$

$$\frac{\mathrm{d}}{\mathrm{du}}\mathrm{shu} = \mathrm{chu}\ \mathrm{dhu}$$

$$\frac{\mathrm{d}}{\mathrm{du}}\mathrm{dhu} = \mathrm{k}^2\mathrm{shu}\ \mathrm{chu}$$

With the addition theorems above and for a given k with 0<k<1, the Jacobi hyperbolic functions therefore are the solutions to the following non-linear ordinary differential equations.

(i) sh (x) solves the differential equations

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} - (1 + \mathrm{k}^2)\mathrm{y} - 2\mathrm{k}^2\mathrm{y}^3 = 0$$

and

$$\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2 = (1 + \mathrm{y}^2)(1 + \mathrm{k}^2\mathrm{y}^2)$$

(ii) ch (x) solves the differential equations

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + (2\mathrm{k}^2 - 1)\mathrm{y} - 2\mathrm{k}^2\mathrm{y}^3 = 0$$

and

$$\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2 = (\mathrm{y}^2 - 1)(1 - \mathrm{k}^2 + \mathrm{k}^2\mathrm{y}^2)$$

(iii) dh (x) solves the differential equations

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + (2 - \mathrm{k}^2)\mathrm{y} - 2\mathrm{y}^3 = 0$$

and

$$\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2 = \mathrm{k}^4\,\mathrm{sh}^2\mathrm{u}\,\mathrm{ch}^2\mathrm{u}$$

4.Conclusions

The result of above study indicates that Jacobi hyperbolic functions are analogous to trigonometric and Jacobi elliptic functions. They have fundamental importance in application of differential equations as they are solutions of non-linear ordinary differential equations. Jacobi hyperbolic functions open up a window of solvable non-linear differential equations, all of which relate to physical problem and physical phenomenon.

Acknowledgments

I dedicated my work to my husband for his full support.

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