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Global Analysis of Modified SIR Epidemic Model with Immigration and Non-Monotonic Incidence Rate Under Treatment

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ABSTRACT

An epidemic model with non-monotonic incidence rate under a limited resource for treatment is proposed to understand the effect of the capacity of the treatment. This model is investigated and analyzed by Kar and Batabyal. Also this model is modified by Gajendra et. al. We have reinvestigated the model by considering modified non-monotonic incidence rate under treatment. Let us assume that the treatment rate is proportional to the number of infective when it is below the capacity and is a constant when the number of infective is larger than the capacity. Existence and stability of the disease free and endemic equilibrium are investigated for both the cases. Some numerical simulations and analysis are given to illustrate the analytical results.

KEYWORDS : Epidemiology, Treatment function, Basic reproductive number, Endemic, Stability

1. Introduction

Mathematical modeling is an important tool to understand and predict the spread of infectious diseases. In this process, rate of incidence plays a crucial role. The incidence is an epidemiological model is the rate at which susceptible become infectious. V. Capasso and Serio [2] introduced a saturated incidence rate g(I)S into epidemic models, where g(I) tends to a saturation level when I gets large, i.e., $g(I) = kI / (1+aI)$. J. Mena Lorca and Hethcote [7] also analysed an SIRS model with the same saturation incidence. Several different incidence rates have been proposed by other researchers. Nonlinear incidence rate of the form bI^pS^q were investigated by Liu. et al. [12]. A very general form of non-linear incidence rate was considered by Derrick and Driessche [3]. Ruan and Wang [14] studied an epidemic model with a specific non linear incidence rate $kI^2 S / (1+aI^2)$ and presented a detailed qualitative and bifurcation analysis of the model . A more general incidence $\lambda I^pS / (1+aI^q)$ was proposed by many other researchers [1,5,8,11,13]. Xiao and Ruan [17] proposed an epidemic model with non-monotonic incidence rate $\lambda IS / (1+aI^2)$. Besides the rate and nature of incidence, treatment plays an important role to control the spread of diseases. Wang [14] proposed a treatment function

$$T(I) = \begin{cases} rI & ; \quad if \quad 0 \le I \le I_0, \\ K_1 & ; \quad if \quad I > I_0 \end{cases} \qquad (1)$$

where $K_1 = rI_0$ for some fixed value I_0. Kar and Batabyal [10] proposed a SIR model with non-monotonic incidence rate suggested by Xiao and Ruan [17] incorporating the above treatment function and non-monotonic incidence rate under a treatment function.

2. The Mathematical Model

Following Gajendra et. al. [6], the proposed model is

(2)

$$\frac{dS}{dt} = a - dS - \frac{\lambda SI}{1+\alpha_1 I+\alpha_2 I^2} + \beta R + \mu$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha_1 I+\alpha_2 I^2} - (d+m)I - T(I)$$

$$\frac{dR}{dt} = mI - (d+\beta)R + T(I)$$

where $S(t)$, $I(t)$, $R(t)$ denote the number of susceptible, infective, recovered individuals, respectively. a is the recruitment rate of the population, d is the natural death rate of the population, λ is the proportionality constant, m is the natural recovery rate of the infective individuals, β is the rate at which recovered individuals lose immunity and return to susceptible class, μ the increase of susceptible at a constant rate and α_1 and α_2 are the parameter measures of the psychological or inhibitory effects and all other parameters have the same meaning as given in [10]. In this work take the treatment function $T(I)$ defined by

$$T(I) = \begin{cases} rI & ; \quad if \quad 0 < I \le I_0, \\ K_1 & ; \quad if \quad I > I_0 \end{cases}$$

This means that the treatment rate is proportional to the infective when the number of infective is less or equal to some fixed value I_0 and the treatment is constant when the number of infective crosses the fixed value I_0 In the next section we study the stability of the model taking two different cases of treatment function. Now we shall study the property of these equilibria and perform a global qualitative analysis of model (2).

3. Mathematical Analysis

Case (i) : SIR model with $0 \le I \le I_0$

When $T(I) = rI$, our model reduces to

$$\frac{dS}{dt} = a - dS - \frac{\lambda SI}{1+\alpha_1 I+\alpha_2 I^2} + \beta R + \mu$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha_1 I+\alpha_2 I^2} - (d+m)I - rI$$

$$\frac{dR}{dt} = mI - (d+\beta)R + rI \qquad (3)$$

When we put the time derivatives equal to zero, we get a disease free equilibrium $E_0(a+\mu/d,0,0)$. For the endemic equilibrium $E^*(S^*, I^*, R^*)$ is the solution of

$$a - dS - \frac{\lambda SI}{1+\alpha_1 I+\alpha_2 I^2} + \beta R + \mu = 0 \qquad (4)$$

$$\frac{\lambda IS}{1+\alpha_1 I+\alpha_2 I^2} - (d+m+r)I = 0 \qquad (5)$$

$$(m+r)I - (d+\beta)R = 0 \qquad (6)$$

From equation (6), we get

$$R = \frac{(m+r)I}{d+\beta} \quad \text{and} \quad I = \frac{R(d+\beta)}{m+r}$$

Using this result in (5), we have

$$S = \frac{(d+m+r)(1+\alpha_1 I+\alpha_2 I^2)}{\lambda}$$

Using this in (4), we get

$$d(d+m+r)\alpha_2 I^2 + \left[d(d+m+r)\alpha_1 + (d+m+r)\lambda - \frac{\beta\lambda(m+r)}{d+\beta}\right] I - (a+\mu)\lambda + d(d+m+r) = 0$$

$$I = -\frac{\left((d+m+r)(d\alpha_1+\lambda) - \frac{\beta\lambda(m+r)}{d+\beta}\right)}{2\alpha_2 d(d+m+r)}$$

$$+\sqrt{\frac{\left((d+m+r)(d\alpha_1+\lambda) - \frac{\beta\lambda(m+r)}{d+\beta}\right)^2 - 4\alpha_2 d(d+m+r)[d(d+m+r)-(a+\mu)\lambda}{4\alpha_2^2 d^2 (d+m+r)^2}}$$

We define the basic reproductive number

$$R_0 = \frac{\lambda(a+\mu)}{d(d+m+r)}$$

A unique positive equilibrium *E* (S*, I*, R*)*, where *S*, R** and *I** are given by

$$S^* = \frac{1}{d}\{(a+\mu) - (d+m+r-\frac{\beta(m+r)}{d+\beta})I^*\}$$

$$I^* = \frac{-\lambda(d+m+r-\frac{\beta(m+r)}{d+\beta}) + \sqrt{\Delta}}{2\alpha_2 d(d+m+r)} \quad \text{and} \quad R^* = \frac{(m+r)I^*}{d+\beta}$$

$$where \Delta = \lambda^2\{(d+m+r)(1-\frac{d\alpha_1}{\lambda}) - \frac{\beta(m+r)}{d+\beta}\}^2 - 4\alpha_2 d^2(d+m+r)^2[1-R_0]$$

To study the dynamics of model (3), we first present a lemma.

Lemma 3.1 *The plane S(t) + I(t) + R(t) = (a + μ)/d is an invariant manifold of the system(3), which is attracting in the first octant.*

Proof. Summing up the three equations in (3) and denoting $N(t) = S(t) + I(t) + R(t)$, we have

$$\frac{dN}{dt} = (a+\mu) - dN \tag{7}$$

This implies that *N(t) = a+μ/d* is a solution of equation (7) and for any $N(t_0) \geq 0$, the general solution of equation (7) is

$$N(t) = \frac{1}{d}\{[a+\mu] - [(a+\mu) - dN(t_0)]e^{-d(t-t_0)}\}$$

Thus, $N(t) \to a+\mu/d$ as limit $t \to \infty$ which implies the conclusion.We consider the reduced system as

$$\frac{dI}{dt} = \frac{\lambda I\left(\frac{a+\mu}{d} - I - R\right)}{1+\alpha_1 I + \alpha_2 I^2} - (d+m+r)I \triangleq P(I,R)$$

$$\frac{dR}{dt} = (m+r)I - (d+\beta)R \triangleq Q(I,R) \tag{8}$$

We have the following result regarding the nonexistence of periodic orbits in system (8), which implies the nonexistence of periodic orbits of system (3) by Lemma 3.1.

Theorem 3.2 *System (8) does not have nontrivial periodic orbits.*

Proof. Consider system (8) for *I > 0* and *R > 0*. Take a Dulac function

$$D(I,R) = \frac{1+\alpha_1 I + \alpha_2 I^2}{\lambda I}$$

We have

$$\frac{\partial(DP)}{\partial I} + \frac{\partial(DQ)}{\partial R} = -1 - \frac{2\alpha_2(d+m+r)}{\lambda} I - \frac{1+\alpha_1 I + \alpha_2 I^2}{\lambda I}(d+\beta) R < 0$$

The conclusion follows:

To study the properties of the disease-free Equilibrium E_0 and the endemic equilibrium E^*, we rescale the system (8) by

$$x = \frac{\lambda}{d+\beta} I, y = \frac{\lambda}{d+\beta} R, T = (d+\beta)t$$

Now we get

$$\frac{dx}{dT} = \frac{x}{1+v_1 I + v_2 I^2}(K - x - y) - ux$$

$$\frac{dy}{dT} = wx - y \tag{9}$$

where

$$K = \frac{\lambda(a+\mu)}{d(d+\beta)}, u = \frac{d+m+r}{d+\beta}, v_1 = \frac{\alpha_1(d+\beta)}{\lambda}, v_2 = \frac{\alpha_2(d+\beta)^2}{\lambda^2} w = \frac{m+r}{d+\beta}$$

Observation and simple calculation shows that $E_0(0,0)$ is the disease free equilibrium and the unique positive equilibrium *(x*, y*)* of the system (9) is the endemic equilibrium *E** of the model system (3). It exists if $u - K < 0$ and *x** is a positive root of the equation

$$uv_2 x^{*2} + (1 + w + uv_1)x^* + (u - K) = 0 \tag{10}$$

and $y^* = wx^*$ (11)

Equation (10) gives

$$x^* = \frac{-(1+w+uv_1) + \sqrt{(1+w+uv_1)^2 - 4uv_2(u-K)}}{2uv_1} \tag{12}$$

Now determine the stability and topological type of *(0, 0)*. The Jacobian matrix of system (9) at *(0, 0)* is

$$M_0 = \begin{bmatrix} K-u & 0 \\ w & -1 \end{bmatrix}$$

Theorem 3.3 *The disease free equilibrium (0, 0) of system (9) is*

(i) a stable hyperbolic node if u-K>0;
(ii) a saddle-node if u-K=0;
(iii) a hyperbolic saddle if u-K<0.

When *u-K<0,* we discuss the stability and topological type of the endemic equilibrium (x^*, y^*). The Jacobian matrix of (9) at (x^*, y^*) is

$$M_1 = \begin{bmatrix} \frac{x^*\{v_2 x^* + (v_1 + 2v_2 x^*)w \quad * - K(v_1 + 2v_2 x^*) - 1\}}{(1+v_1 x^* + v_2 x^{*2})^2} & \frac{-x^*}{1+v_1 x^* + v_2 x^{*2}} \\ w & -1 \end{bmatrix}$$

$$\det(M_1) = \frac{x^*\{1 + w + K(v_1 + 2v_2 x^*) - v_2(1+w)x^{*2}\}}{(1+v_1 x^* + v_2 x^{*2})^2}$$

The sign of *det* (M_1) is determined by

$$P_1 = -v_2(1+w)x^{*2} + K(v_1 + 2v_2 x^*) + (1+w) \tag{13}$$

Note that $uv_2 x^{*2} + (1+w)x^* + u - K = 0$.

Using (11), (12) in (13), we get

$$P_1 = \frac{\Delta_1(1+w)}{2u^2 v_2}\left[(1+w+uv_1) + \frac{2Kuv_2}{1+w} - \Delta_1\right]$$

Here

$$\left[\left\{(1+w+uv_1) + \frac{2Kuv_2}{1+w}\right\}^2 - \Delta_1^2\right] > 0 \Rightarrow \left[(1+w+uv_1) + \frac{2Kuv_2}{1+w} - \Delta_1\right] > 0$$

This implies $P_1 > 0$. Hence, *det* $(M_1) > 0$ and (x^*, y^*) is a node or a focus or a center. Furthermore, we have the following result on the stability of (x^*, y^*).

Theorem 3.4 *Suppose* $u - K < 0$, *then there is a unique endemic equilibrium* (x^*, y^*) of model (9), which is a stable node.

Proof. We know that the stability of (x^*, y^*) is determined by *trace(M$_1$)*. We have

$$Trace\ (M_1) = \frac{x^*\{v_2x^* + (v_1 + 2v_2x^*)wx^* - K(v_1 + 2v_2x^*) - 1\}}{(1 + v_1x^* + v_2x^{*2})^2} - 1$$

The sign of *trace (M$_1$)* is determined by

$$P_2 = -v_2^2v^{*4} + (1+2w-2v_1)v_2x^{*3} + (v_1w - v_1^2 - 2Kv_2 - 2v_2)x^{*2} - (Kv_1 + 2v_1 + 1)x^* - 1 \qquad (14)$$

We claim that $P_2 \neq 0$. To see this, note that

$$uv_2x^{*2} + (1+w)x^* + u - K = 0.$$

Then we have

Let $\xi = uv_2x^{*2} + (1+w)x^* + u - K$

Using again equation (10) in (14), we get $P_2 = P_3x^* - P_4$ where

$P_3 = (1+w+uv_1)\{(1+w)^2 + u(1+w)(1+2w+uv_1) + 2uv_2K(1+w)\} + u^2v_2K(1+2w) + 2u^2v_2(1+w)(K-u) + u^3v_1v_2K$

$P_4 = (K-u)\{(1+w)^2 + u(1+w)(1+2w+uv_1)\} + uv_2\{(K-u)^2 + 2u(1+K)(K-u) + u^2\}$

Therefore both P_3 and P_4 are positive for any set of parameters with $K > u$. So when the point (x^*, y^*) exists, it is locally stable if $x^* < P_4/P_3$. Assume that $S_2 = 0$. Since $\xi = 0$, it follows that $P_3 = 0$. When $u\text{-}K < 0$, we have $S_3 > 0$. Therefore, $S_2 \neq 0$ for any positive value of the parameters v_1, v_2, w and K, that is *trace(M$_1$)*$\neq 0$. Thus, $u\text{-}K < 0$ implies that (x^*, y^*) does not change stability. Take $u=1$, $K=2$, $v_1=2$, $v_2=1$, $w=1$. Then $x^* = -2+\sqrt{5}$, $y^* = -2+\sqrt{5}$, *Trace(M$_1$)* = -2.6758 < 0. This completes the proof.

Summarizing Theorems 3.2 - 3.4, we have the following results on the dynamics of the original model (3).

Theorem 3.5 *Let R$_0$ be defined by*

(i) If $R_0 < 0$, then model (3) has a unique disease-free equilibrium $E_0 = (a+\mu/d, 0, 0)$, which is a global attractor in the first octant.
(ii) If $R_0 = 1$, then model (3) has a unique disease-free equilibrium $E_0 = (a+\mu/d, 0, 0)$, which *attracts all orbits in the interior of the first octant.*
(iii) If $R_0 > 1$, then model (3) has two equilibria, a disease-free equilibrium $E_0 = (a+\mu/d, 0, 0)$ and *an endemic equilibrium E* = (S*, I*, R*). The endemic equilibrium E* is a global attractor in the interior of the first octant.*

Case 2 : SIR model with $I > I_0$.

In this case the model is

$$\begin{aligned}
\frac{dS}{dt} &= a - dS - \frac{\lambda SI}{1 + \alpha_1 I + \alpha_2 I^2} + \beta R + \mu \\
\frac{dI}{dt} &= \frac{\lambda IS}{1 + \alpha_1 I + \alpha_2 I^2} - (d+m)I - K_1 \\
\frac{dR}{dt} &= mI - (d+\beta)R + K_1
\end{aligned} \qquad (15)$$

Since $S + I + R = (a+\mu)/d$ is invariant manifold of the system, the model reduces to

$$\frac{dI}{dt} = \frac{\lambda I(\frac{a+\mu}{d} - I - R)}{1 + \alpha_1 I + \alpha_2 I^2} - (d+m)I - K_1 \qquad (16)$$

$$\frac{dR}{dt} = mI - (d+\beta)R + K_1 \qquad (17)$$

Rescaling above equation, we get,

$$\frac{dx}{dT} = \frac{x(K - x - y)}{1 + v_1I + v_2I^2} - u_1x - c \qquad (18)$$

$$\frac{dy}{dT} = w_1x - y + c \qquad (19)$$

where

$$K = \frac{\lambda(a+\mu)}{d(d+\beta)},\ u_1 = \frac{d+m}{d+\beta},\ v_1 = \frac{\alpha_1(d+\beta)}{\lambda},\ v_2 = \frac{\alpha_2(d+\beta)^2}{\lambda^2},\ c = \frac{\lambda K_1}{(d+\beta)^2},\ w_1 = \frac{m+r}{d+\beta}$$

For the equilibrium we put

$$\frac{x(K - x - y)}{1 + v_1I + v_2I^2} - u_1x - c = 0 \quad and \quad w_1x - y + c$$

Put $y = w_1x + c$ in the first equation we get

$$u_1v_2x^3 + (1+w+cv_2+u_1v_1)x^2 + (c+u_1+cv_1-K)x + c = 0 \qquad (20)$$

If $c + u_1 + cv_1 > K$, equation (20) has no positive solution. But if $c + u_1 + cv_1 < K$, then it has either two positive roots or no positive roots. From the theory of equations

$$p_0x^3 + 3p_1x^2 + 3p_2x + p_3 = 0 \qquad (21)$$

has all roots are real if the conditions $G^2 + 4H^3 < 0$ and $H < 0$, where $H = p_0p_2 - p_1^2$ and

$G = p_0^2p_3 - 3p_0p_2p_1 + 2p_1^3$ are satisfied.

Comparing (20) and (21) we get

$p_0 = u_1v_2$, $p_1 = (1+w+cv_2+u_1v_1)/3$, $p_2 = (c+u_1+cv_1-K)/3$ and $p_3 = c$.

Now

$$H = p_0p_2 - p_1^2 = u_1v_2[\frac{(u_1 + c - K + u_1v_1)}{3}] - [\frac{(1 + w_1 + cv_2 + u_1v_1)}{3}]^2$$

When $c + u_1 + cv_1 < K$ then $H < 0$ and

$$G^2 + 4H^3 = (p_0^2p_3 - 3p_0p_2p_1 + 2p_1^3)^2 + 4(p_0p_2 - p_1^2)^3$$

$$= p_0^2(p_0^2p_3^2 - 6p_0p_2p_1p_3 + 4p_3p_1^3 + 4p_0p_2^3 - 3p_1^2p_2^2)$$

To fulfill the condition $G^2 + 4H^3 < 0$ we must have

$$p_0^2p_3^2 + 4p_3p_1^3 + 4p_0p_2^3 < 3p_1^2p_2^2 + 6p_0p_1p_2p_3 \qquad (22)$$

The variation matrix for the system (18) - (19) at the equilibrium point (,)is

$$M_2 = \begin{bmatrix} \frac{\{(1+v_1\bar{x}+2v_2\bar{x^2})(K-2\bar{x}-\bar{y}) - (v_1+2v_2\bar{x})(K\bar{x}-\bar{x^2}-\bar{xy})\}}{(1+v_1\bar{x}+v_2\bar{x^2})^2} - u_1 & \frac{-\bar{x}}{1+v_1\bar{x}+v_2\bar{x^2}} \\ w_1 & -1 \end{bmatrix}$$

$det(M_2) =$

$$\frac{1}{(1+v_1\bar{x}+v_2\bar{x^2})^2}\left\{\begin{array}{l} -(1+v_1\bar{x}+2v_2\bar{x^2})(K-2\bar{x}-\bar{y}) + (v_1+2v_2\bar{x})(K\bar{x}-\bar{x^2}-\bar{xy}) \\ +u_1(1+v_1\bar{x}+v_2\bar{x^2})^2 + w_1\bar{x}(1+v_1\bar{x}+v_2\bar{x^2}) \end{array}\right\}$$

Solving this we get the sign of the *det (M$_2$)* is governed by the factor

$$p_5 = u_1v_2^2\,\bar{x}^4 + 2u_1v_1v_2\,\bar{x}^3 + (Kv_2 - cv_2 + 2u_1v_2 + u_1v_1^2 + v_1w_1 + v_1)\,\bar{x}^2 + (2+2w_1+2u_1v_1)\,\bar{x} + (c+u_1-K) \qquad (23)$$

We recall equation (20) and use it in (23). After some calculations we get,

$$u_1p_5 = [\{2u_1v_2(K-c) + u_1^2v_2 + u_1v_1(1+w_1-cv_2+u_1v_1)\} + \{(1+w_1)(1+w_1+cv_2+u_1v_1)$$

$$+(cv_2 - u_1v_1)(1+w_1+cv_2+u_1v_1)\}]\,\bar{x}^2 + [(2u_1 + 2u_1w_1 - cu_1v_2 + 2u_1^2v$$

$+(1+w_1+cv_2-u_1v_1)(u_1+c-K+cv_1)]\bar{x} + [cu_1 + u_1^2 - Ku_1 - cu_1v_1 + c(1+w_1+cv_2)]$ (24)

(or) $u_1 p_5 = \xi_1 \bar{x}^2 + \xi_2 \bar{x} + \xi_3$

where

$\xi_1 = u_1^2 v_2 + 2u_1v_2(K-c) + u_1v_1(1+w_1) + (1+w_1)(1+w_1+cv_2+u_1v_1) + (cv_2 - u_1v_1)(1+w_1+cv_2)$

Here $\xi_1 > 0$ if $K > c$ and $cv_2 \geq u_1 v_1$

Hence with the above condition, sufficient condition for P_5 to be positive if

$$\xi_2^2 - 4\xi_1\xi_3 \leq 0 \quad (25)$$

$$Trace(M_2) = \frac{\left\{(1+v_1\bar{x}+2v_2\bar{x}^2)(K-2\bar{x}-\bar{y}) - (v_1+2v_2\bar{x})(K\bar{x}-\bar{x}^2-\bar{x}\bar{y})\right\} - (u_1+1)(1+v_1\bar{x}+v_2\bar{x}^2)^2}{(1+v_1\bar{x}+v_2\bar{x}^2)^2}$$

Solving we get the sign of $Trace(M_2)$ is determined by the sign of

$p_6 = -(u_1+1)v_2^2\bar{x}^4 + [w_1 - 2v_1(u_1+1)]v_2\bar{x}^3 + [(c-K)v_2 - v_1 - (u_1+1)(v_1^2+2v_2)]\bar{x}^2 - [(2+w_1) + (u_1+1)2v_1]\bar{x} + (K-c) - (u_1+1)$ (26)

This equation can be written using equation (20), as

$u_1^2 p_6 = \eta_1\bar{x}^2 + \eta_2\bar{x} + \eta_3$ where

$\eta_1 = -[(1+w_1+cv_2+u_1v_1)\{(1+w_1)+(cv_2-u_1v_1)+u_1+2u_1w_1+u_1(cv_2-u_1v_1)\}]$

$- [u_1v_2\{(2u_1+1)(K-c)+u_1^2+u_1\}+u_1^2v_1^2(u_1+1)+u_1^2v_1]$

$\eta_2 = u_1[cv_2 u_1 + cv_2v_1 - 2u_1 - w_1 u_1 - 2u_1(u_1+1) v_1] - (u_1+c+cv_1-K)[2u_1w_1+2u_1v_2$

$+u_1+w_1+cv_2+1-u_1^2v_1-u_1v_1]$

$\eta_3 = u_1^2(K-c-u_1-1) - c[2u_1w_1 + cu_1v_2 + u_1 + w_1 + cv_2 - u_1^2v_1 - u_1v_1]$

Here $\eta_1 < 1$ for $K > c + u_1 + cv_1$ and $u_1v_1 \leq cv_2$.

Therefore the sufficient condition for $p_6 < 0$ is

$$\eta_2^2 - \eta_1\eta_3 \leq 0 \quad (27)$$

We conclude the above finding in the following theorem.

Theorem 3.5 *When $K > u_1+c_1+cv_1$ and $u_1v_1 \leq cv_2$ the system has two positive equilibrium points*

$(\bar{x_1}, \bar{y_1})$ and $(\bar{x_2}, \bar{y_2})$, where $\bar{x_1}$ and $\bar{x_2}$ are the solution of the equation (20) under the parametric conditions given by (22) and when the conditions (25) and (27) are fulfilled at some equilibrium point that point must be asymptotically stable.

4. Numerical Simulations

Case (i) : When the treatment rate is ∞ to the infective so that $0 \leq I \leq I_0$. We choose the parameters in the model as follows.

$a = 3, d = 0.1, \lambda = 0.4, \mu = 0.2, m = 0.01, r = 0.2$

Here the basic reproduction number $R_0 = 41.29 > 1$.

When $a = 15, d = 2.5, m = 10, r = 0.01, \mu = 0.2, \lambda = 0.4$

We have $R_0 = 0.1944 < 1$. In this case the disease dies out. Consider

$a = 3, d = 0.1, \alpha_1 = 0.5, \alpha_2 = 0.5, \beta = 0.1, \mu = 0.2, \lambda = 0.3, m = 0.01, r = 0.2$

By rescaling the system we see that $u\text{-}K < 1$ hence there exists the unique positive equilibrium $x^*=8.504$ and $y^*=8.929$. For the above parameters $p_4/p_3=9.5798$ and then sufficient condition for stability $x^* < p_4/p_3$ is satisfied. Hence the point is locally stable and it can be pointed out that the values of x^* and y^* given in [9] are interchanged due to misprint however it does not affect the analysis.

Case (ii) : When $I > I_0$, we choose the parameters

$a = 2.5, d = 0.05, \alpha_2 = 3, \beta = 0.15, \lambda = 0.4, m = 0.01, K_1 = 0.7, \mu = 0.2$ with $\alpha_1 = 1$

In this case $S+I+R=(a+\mu)/d = 54$ is invariant manifold. The system reduces to

$$\frac{dI}{dt} = \frac{0.4I(54-I-S)}{1+I+3I^2} - (0.36)I - 0.7$$

$$\frac{dR}{dt} = (0.31)I - (0.20)R + 0.7$$

The rescaling of the parameter gives rise to

$$\frac{dx}{dT} = \frac{x(108-x-y)}{1+0.5I+0.751^2} - 1.8x - 7$$

$$\frac{dy}{dT} = 1.55x - y + 7$$

Here $(u\text{-}K) < 0$, and hence there exists unique positive equilibrium point (x^*, y^*) where $x^* = 8.504$ and $y^*=8.929$. For the above choice of parameters $p_3 = 404.02 > 0$, $p_4=3870.44$, $p_4/p_3 = 9.5798$ and therefore the sufficient condition for local stability i.e., $x^* < p_4/p_3$ is satisfied.

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