Volume : 2 | Issue : 5 | May 2013 • ISSN No 2277 - 8160

Research Paper | Mathematics

SOME CHARACTERISTICS OF DERIVATIONS ON Γ–BANACH ALGEBRAS

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ABSTRACT

In this paper, we establish a correspondence between derivations on a Γ-Banach algebra and derivations on its right (and left) operator Banach algebras. Let M be a Γ-Banach algebra and R and L be respectively the right and left operator Banach algebras of M. We have shown that for a given derivation D on M, there exists a derivation on R, and also on L. Different properties of this derivation are discussed here. The converse of this result is also established for the Γ-Banach algebra M satisfying some specific conditions. Finally, we construct a derivation Dγ for the projective tensor product R⊗γL and investigate some interesting properties of this derivation.

KEYWORDS: Γ- algebra, derivation, tensor product, operator Banach algebra

1. Introduction.

The significant development of the theory of derivations started around 1955. At the present times, many technical results relating derivations with automorphisms, hermitian operators, normal operators etc., are developed by various prominent research workers (refer to [2], [5], [10], [13]). In 2002, Y. Ceven ([5]) defined a Jordan left derivation on Gamma rings. He established that Jordan left derivation on completely prime Gamma rings is a left derivation. In 2007, M. Asci and S. Ceran ([1]) obtained some commutativity results on prime Gamma rings with left derivation. In 2011, Paul and Rahman ([15]) established some interesting results on Jordan left derivations of semiprime Gamma rings. In 2012, Dey, Paul and Rakhimov ([6]) studied generalized derivations in semiprime Gamma rings.

In this paper, we study derivations on the operator Banach algebras of a Gamma Banach algebra. Let *M* be a Γ-Banach algebra and *R* and *L* be respectively the right and left operator Banach algebras of *M*. From a given derivation *D* on *M*, we construct a derivation $\tilde{D}$ on *R* (and also on *L*). Different properties of this derivation $\tilde{D}$ are established here. Conversely, for a given derivation $\tilde{D}$ on *R* (or *L*), we have constructed the derivation *D* for the Γ-Banach algebra *M* satisfying the condition $x\Gamma M=0 \Rightarrow x=0$ (or, $M\Gamma x=0 \Rightarrow x=0$). Finally we construct a derivation on the projective tensor product $R\hat{\otimes}_\gamma L$ and discuss some of its properties.

Before discussing the main results, we first present some preliminary definitions (refer to [2], [3], [4], [6], [14]).

Definition 1.1 Let *M* and Γ be two linear spaces over a field *F*. *M* is said to be a Γ-algebra over *F* if, for $x, y, z \in M$; $\alpha, \beta \in \Gamma$; $a \in F$, the following conditions are satisfied:

(i) $x\alpha y \in M$,
(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,
(iii) $a(x\alpha y) = (ax)\alpha y = x(a\alpha)y = x\alpha(ay)$
(iv) $x\alpha(y+z) = x\alpha y + x\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $(x+y)\alpha z = x\alpha z + y\alpha z$

The G-algebra is denoted by (M, G). If M and G are normed linear spaces over F, then G-algebra (M, G) is called a G-normed algebra if conditions (i)-(iv) hold and further

(v) $\|x\alpha y\| \le \|x\|\|\alpha\|\|y\|$ holds.

A Γ-normed algebra (M, Γ) is called a Γ-Banach algebra if *M* is a Banach space. Any Banach algebra can be regarded as a Γ-Banach algebra by suitably choosing Γ.

Definition 1.2 Let *M* be a Γ-Banach algebra. For $\alpha \in \Gamma$, an additive mapping $D: M \to M$ is called an α- derivation on *M* if $D(x\alpha y)=(Dx)\alpha y + x\alpha(Dy)$ for all $x, y \in M$.

D is called a derivation if it is an α-derivation for all $\alpha \in \Gamma$.

D is called a generalized derivation if there exists a derivation $d: M \to M$ such that $D(x\alpha y)=(Dx)\alpha y + x\alpha(dy)$ for all $x \in M$ and $\alpha \in \Gamma$. An additive mapping *D* on *M* is said to be a Jordan derivation if $D(x\alpha x)=(Dx)\alpha x + x\alpha(Dx)$ for all $x, y \in M$ and $\alpha \in \Gamma$. *D* is said to be a Jordan generalized derivation if there exists a derivation $d: M \to M$ such that $D(x\alpha x)=(Dx)\alpha x + x\alpha(dx)$ for all $x \in M$ and $\alpha \in \Gamma$.

Definition 1.3 Let *M* be a G-Banach algebra and let $x \in M$, $a \in G$. Then the mapping $[a, x]$ defined by $y[a, x] = yax$ "$y \in M$ is a right Banach space endomorphism of *M*. The collection *R* of all endomorphisms generated by $[a, x]$; $a \in G$, $x \in M$ is a Banach algebra under the operations:

$[\alpha, x] + [\alpha, y] = [\alpha, x+y]$,

$[\alpha, x] + [\beta, x] = [\alpha+\beta, x]$,

$a[\alpha, x] = [\alpha, ax] = [a\alpha, x]$ where $a \in F$, $[\alpha, x][\beta, y] = [\alpha, x\beta y]$, $\alpha, \beta \in \Gamma$, and the norm $\|[\alpha, x]\| = \|\alpha\|_\Gamma . \|x\|_M$.

This Banach algebra is termed as the right operator Banach algebra of Γ-Banach algebra *M*. We can similarly define the left operator Banach algebra *L* of *M* as the Banach algebra generated by the set of all left endomorphisms of *M* in the form $[x, \alpha]$ where $[x, \alpha]y = x\alpha y$ "$y \in M$.

Definition 1.4 Let *X* and *Y* be two normed spaces. The *projective tensor norm* $\|\ .\ \|_\gamma$ on $X \otimes Y$ is defined as:

$\|u\|_\gamma = \inf\{\sum_i \|x_i\|\|y_i\| : u = \sum_i x_i \otimes y_i\}$ where the infimum is taken over all (finite) representations of *u*. The completion of $(X \otimes Y, \|\ .\ \|_\gamma)$ is called the projective tensor product of *X* and *Y*, and is denoted by $X \hat{\otimes}_\gamma Y$.

2. Main Results

First, we consider the right operator Banach algebra *R* of the Γ-Banach algebra *M* and proceed to construct a derivation here.

Theorem 2.1. Let *D* be a derivation on the Γ-Banach algebra *M*. On the right operator Banach algebra *R* of *M*, we define the operator $\tilde{D}$ by:

$$\tilde{D}(\sum_i [\alpha_i, x_i]) = \sum_i [\alpha_i, D(x_i)]$$

Then $\tilde{D}$ is a derivation on *R* and $\|\tilde{D}\| \le \|D\|$ $\tilde{D}$ is continuous if *D* is continuous. Moreover, if *D* is a generalized derivation, then $\tilde{D}$ is also a generalized derivation.

Proof. Since $D: M \to M$ is a derivation, so, $D(x\alpha y)=(Dx)\alpha y + x\alpha(Dy)$ for all $x, y \in M$, $\alpha \in \Gamma$.

Clearly, $\tilde{D}: R \to R$ defined by

$\tilde{D}(\sum_i [\alpha_i, x_i]) = \sum_i [\alpha_i, D(x_i)]$, for each $\sum_i [\alpha_i, x_i] \in R$, is well defined and linear.

For $p=\sum_i [\alpha_i, x_i]$ and $q=\sum_j [\beta_j, y_j]$ in *R*,

$\tilde{D}(pq) = \tilde{D}(\sum_{i,j}[\alpha_i, x_i\beta_j y_j])$

$= \sum_{i,j}[\alpha_i, D(x_i\beta_j y_j)]$

$= \sum_{i,j}[\alpha_i, D(x_i)\beta_j y_j + x_i\beta_j D(y_j)]$

$= \sum_{i,j}[\alpha_i, D(x_i)\beta_j y_j] + \sum_{i,j}[\alpha_i, x_i\beta_j D(y_j)]$

$= \sum_i [\alpha_i, D(x_i)] \sum_j [\beta_j, y_j] + \sum_i [\alpha_i, x_i] \sum_j [\beta_j, D(y_j)]$

$= \tilde{D}(p)q + p\tilde{D}(q)$

Thus, $\tilde{D}$ is a derivation on *R*.

Let *D* be continuous. So, for the sequence $\{x_n\}$ in *M* converging to *x*, we have, $Dx_n \to Dx$. Let $\{[\alpha_n, x_n]\}$ be a sequence in R such that $\alpha_n \to \alpha$ and $x_n \to x$, so that $[\alpha_n, x_n] \to [\alpha, x]$.

Now, $\|\tilde{D}([\alpha_n, x_n]) - \tilde{D}([\alpha, x])\|$

$= \|[\alpha_n, D(x_n)] - [\alpha, D(x)]\|$

$\le \|[\alpha_n, D(x_n) - D(x)]\| + \|[\alpha_n - \alpha, D(x)]\|$

$\le \|\alpha_n\| \|D(x_n) - D(x)\| + \|\alpha_n - \alpha\| \|D(x)\|$

$\to 0$ as $n \to \infty$.

Thus, $\tilde{D}([\alpha_n, x_n]) \to \tilde{D}([\alpha, x])$. So, $\tilde{D}$ is continuous.

Let *D* be a generalized derivation. Then there exists a derivation $d: M \to M$ such that $D(x\alpha y) = (Dx)\alpha y + x\alpha(dy)$ for all $x, y \in M$, $\alpha \in \Gamma$. Now, corresponding to the derivation *d*, we have a derivation $\tilde{d}$ on *R* as defined above.

For $p = \sum_i [\alpha_i, x_i]$ and $q = \sum_j [\beta_j, y_j]$ in *R*,

$\tilde{D}(pq) = \sum_{i,j}[\alpha_i, D(x_i\beta_j y_j)]$

$= \sum_{i,j}[\alpha_i, D(x_i)\beta_j y_j + x_i\beta_j d(y_j)]$

$= \sum_i [\alpha_i, D(x_i)] \sum_j [\beta_j, y_j] + \sum_i [\alpha_i, x_i] \sum_j [\beta_j, d(y_j)]$

$= \tilde{D}(p)q + p\tilde{d}(q)$

Thus, $\tilde{D}$ is a generalized derivation.

Theorem 2.2 Let $D: M \to M$ be a Jordan derivation such that $D(x\alpha y) = 0$ for $x \neq y$. Then

$\tilde{D}(\sum [\alpha_i, x_i] \sum [\alpha_i, x_i])$

$= \sum_i [[\alpha_i, x_i](\tilde{D}[\alpha_i, x_i]) + (\tilde{D}[\alpha_i, x_i])[\alpha_i, x_i]]$.

Proof. $\tilde{D}(\sum_i [\alpha_i, x_i] \sum_i [\alpha_i, x_i])$

$= \tilde{D}(\sum_i [\alpha_i, x_i] \sum_j [\alpha_j, x_j])$

$= \tilde{D}(\sum_{i,j} [\alpha_i, x_i\alpha_j x_j]) = \sum_{i,j} [\alpha_i, D(x_i\alpha_j x_j)]$

$= \sum_i [\alpha_i, D(x_i\alpha_i x_i)]$

$= \sum_i [\alpha_i, D(x_i)\alpha_i x_i + x_i\alpha_i D(x_i)]$

$= \sum_i ([\alpha_i, D(x_i)][\alpha_i, x_i] + [\alpha_i, x_i][\alpha_i, D(x_i)]$

$= \sum_i [[\alpha_i, x_i](\tilde{D}[\alpha_i, x_i]) + (\tilde{D}[\alpha_i, x_i])[\alpha_i, x_i]]$.

Now, we try to find a relationship between the eigen values of *D* and $\tilde{D}$

Theorem 2.3 Let *M* be a Γ-Banach algebra satisfying the condition $M\Gamma x = 0 \Rightarrow x = 0$, $x \in M$. Let *D* be the derivation on *M* and $\tilde{D}$ be the corresponding derivation on *R*. Then $Sp(D) \subseteq Sp(\tilde{D})$. [Sp: Spectrum]

Proof. Let $\lambda \in Sp(D)$. So, $Dx = \lambda x$ for some $x \in M$, $x \neq 0$. For $\alpha (\neq 0) \in \Gamma$, $[\alpha, x] \in R$.

If $[\alpha, x] = 0$, then $y[\alpha, x] = 0 \ \forall\, y \in M$, $\alpha \in \Gamma$

$\Rightarrow y\alpha x = 0 \ \forall\, y \in M$, $\alpha \in \Gamma$

$\Rightarrow M\Gamma x = 0 \Rightarrow x = 0$, which is impossible.

So, $[\alpha, x] \neq 0$.

Now, $\tilde{D}([\alpha, x]) = [\alpha, D(x)] = [\alpha, \lambda x] = \lambda[\alpha, x]$. So, $\lambda \in Sp(\tilde{D})$. Thus, $Sp(D) \subseteq Sp(\tilde{D})$.

We recall that for $\alpha \in \Gamma$, the Γ-Banach algebra *M* is said to be α-prime, if $a\Gamma M\alpha b = 0$ (with $a, b \in M$) $\Rightarrow a = 0$ or $b = 0$ (or, $a\alpha M\Gamma b = 0 \Rightarrow a = 0$ or $b = 0$).

Theorem 2.4 If λ is an eigen value of $\tilde{D}$ with corresponding eigenvector $[\alpha, x]$, and *M* is α-prime ($\alpha \in \Gamma$), then λ is also an eigen value of *D* with eigen vector *x*.

Proof. $\tilde{D}([\alpha, x]) = \lambda[\alpha, x]$

$\Rightarrow [\alpha, D(x)] = [\alpha, \lambda x] \Rightarrow [\alpha, D(x) - \lambda x] = 0$.

So, $y[\alpha, D(x) - \lambda x] = 0 \ \forall\, y \in M$.

Taking $y = m\beta n$, $m, n \in M$, $\beta \in \Gamma$, $m \neq 0$,

$m\beta n\alpha(Dx - \lambda x) = 0$.

Here, $\beta \in \Gamma$ and $n \in M$ are arbitrary. So, $m\Gamma M\alpha(Dx - \lambda x) = 0$.

Since *M* is α-prime, $Dx - \lambda x = 0$, since $m \neq 0$.

Clearly, $x \neq 0$, since $[\alpha, x] \neq 0$. So, λ is an eigen value of *D* with eigen vector *x*.

Analogous results as in the above discussion can also be obtained in case of the left operator Banach algebra *L* of the Γ-Banach algebra *M*.

Theorem 2.5 Let *L* be the left operator Banach algebra of the Γ-Banach algebra *M*. Then

(i) If *D* is a derivation on *M*, then $\tilde{D}: L \to L$ defined by:

$\tilde{D}(\sum_i [x_i, \beta_i]) = \sum_i [D(x_i)\ \beta_i]$ is a derivation on *L*.

(ii) $\|\tilde{D}\| \le \|D\|$ $\tilde{D}$ is continuous, if *D* is continuous.

(iii) If *D* is a generalized derivation, then $\tilde{D}$ is also a generalized derivation.

(iv) If *M* satisfies the condition $x\Gamma M = 0 \Rightarrow x = 0$, $\forall\, x \in M$, then $Sp(D) \subseteq Sp(\tilde{D})$.

(v) If λ is an eigen value of $\tilde{D}$ with corresponding eigen vector $[x, \beta]$, and *M* is β-prime, then λ is also an eigen value of *D* with eigen vector *x*.

Our next aim is to establish the converse of Theorem 2.1, i.e., to construct a derivation *D* on *M* from the derivation $\tilde{D}$ on *R*.

Theorem 2.6 Let $\tilde{D}$ be a derivation on the right operator Banach algebra *R* of the Γ-Banach algebra *M*. Let *M* satisfies the condition $M\Gamma x = 0 \Rightarrow x = 0$. Let $D: M \to M$ be such that $[\alpha, D(x)] = \tilde{D}([\alpha, x])$, $x \in M$. Then

(i) *D* is a derivation on *M*.
(ii) If $\tilde{D}$ is continuous, then *D* is also continuous and $\|D\| \le \|\tilde{D}\|$.
(iii) If $\tilde{D}$ is a generalized derivation, then *D* is also a generalized derivation.
(iv) If $\tilde{D}$ is a Jordan (generalized) derivation, then *D* is also a Jordan (generalized) derivation.

Proof. (i) Clearly, *D* is well defined.

For $x_1, x_2 \in M$, $\alpha \in \Gamma$,

$[\alpha, D(x_1 + x_2)] = \tilde{D}([\alpha, x_1 + x_2])$

$= \tilde{D}([\alpha, x_1]) + \tilde{D}([\alpha, x_2])$

$= [\alpha, D(x_1)] + [\alpha, D(x_2)]$

$\Rightarrow [\alpha, D(x_1 + x_2) - (D(x_1) + D(x_2))] = 0$

$\Rightarrow y\alpha(D(x_1 + x_2) - (D(x_1) + D(x_2))) = 0$ " $y \in M$.

Again α is arbitrary.

Thus, $M\Gamma(D(x_1 + x_2) - (D(x_1) + D(x_2))) = 0$

Now, by the given condition on M, $D(x_1 + x_2) = D(x_1) + D(x_2)$

Similarly, for $a \in F$, $x \in M$, $D(ax) = aD(x)$.

So, D is linear.

For $\alpha, \beta \in \Gamma$ and $x, y \in M$

$[\alpha, D(x\beta y)] = \tilde{D}([\alpha, x\beta y]) = \tilde{D}([\alpha, x][\beta, y])$

$= [\alpha, x]\tilde{D}([\beta, y]) + \tilde{D}([\alpha, x])[\beta, y]$

$= [\alpha, x\beta D(y)] + [\alpha, D(x)\beta y]$

$= [\alpha, x\beta D(y) + D(x)\beta y]$

Now, applying the given condition again, we can show that $D(x\beta y) = D(x)\beta y + x\beta D(y)$.

So, D is a derivation on M.

(ii) Let $\tilde{D}$ be continuous. Let $\{x_n\}$ be a sequence in M converging to x. We take some $\alpha (\neq 0) \in \Gamma$ as fixed. Then the sequence $\{[\alpha, x_n]\}$ in R will converge to $[\alpha, x]$. By the continuity of $\tilde{D}$,

$\tilde{D}([\alpha, x_n]) \to \tilde{D}([\alpha, x])$.

Taking $\varepsilon' = \varepsilon \|\alpha\| > 0$, there exists some positive integer N_0 such that $\|\tilde{D}([\alpha, x_n]) - \tilde{D}([\alpha, x])\| < \varepsilon'$ " $n \geq N_0$.

$\Rightarrow \|[\alpha, D(x_n) - D(x)]\| < \varepsilon'$

$\Rightarrow \|\alpha\| \|D(x_n) - D(x)\| < \varepsilon'$

$\Rightarrow \|D(x_n) - D(x)\| < \varepsilon$ " $n \geq N_0$.

Thus $D(x_n) \to D(x)$. Hence, D is continuous.

Let $\alpha (\neq 0) \in \Gamma$ and $x \in M$.

Now, $\|\tilde{D}([\alpha, x])\| = \|[\alpha, D(x)]\|$

$\Rightarrow \|\alpha\| \|D(x)\| \leq \|\tilde{D}\| \|[\alpha, x]\| = \|\tilde{D}\| \|\alpha\| \|x\|$

$\Rightarrow \|D(x)\| \leq \|\tilde{D}\| \|x\|$ $\Rightarrow \|D\| \leq \|\tilde{D}\|$.

(iii) The proof follows easily.

(iv) Let $\tilde{D}$ be a Jordan derivation.

So, for $p = \sum_i [\alpha_i, x_i] \in R$,

$\tilde{D}(pp) = \tilde{D}(p)p + p\tilde{D}(p)$.

Now, $[\alpha, D(x\alpha x)] = \tilde{D}([\alpha, x][\alpha, x])$

$= \tilde{D}([\alpha, x])[\alpha, x] + [\alpha, x]\tilde{D}([\alpha, x])$

$= [\alpha, D(x)][\alpha, x] + [\alpha, x][\alpha, D(x)]$

$= [\alpha, D(x)\alpha x + x\alpha D(x)]$

So, applying the given condition,

$D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$.

Hence, D is a Jordan derivation.

It is easy to show that if $\tilde{D}$ is a Jordan generalized derivation, then D is also a Jordan generalized derivation.

Remark 2.7 A Γ-Banach algebra M is said to be n-torsion free if $nx = 0$, $x \in M$ implies $x = 0$, where n is a positive integer. M is semiprime if $a\Gamma M\Gamma a = 0 \Rightarrow a = 0$. In 2012, Dey, Paul and Rekhimov (in [6]) proved that if M is a 2-torsion free semiprime Gamma ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ " $a, b, c \in M$, $\alpha, \beta \in \Gamma$ and $D : M \to M$ is a Jordan generalized derivation, then D is a generalized derivation. So, in the above result, if M also satisfies these conditions, then for the Jordan generalized derivation $\tilde{D}$ on R, we get a generalized derivation D on M.

Now, we consider the projective tensor product of right and left operator Banach algebras of the Γ-Banach algebra M. Using the derivation D on M, we construct a derivation D_γ on $R \hat{\otimes}_\gamma L$.

Theorem 2.8 Let M be a Γ-Banach algebra and R and L be respectively the right and left operator Banach algebras of M. Let D be a derivation on M. We define $D_\gamma : R \hat{\otimes}_\gamma L \to R \hat{\otimes}_\gamma L$ by

$D_\gamma(\sum_i p_i \otimes q_i) = \sum_i [\alpha_i, Dx_i] \otimes [y_i, \beta_i]$

$+ \sum_i [\alpha_i, x_i] \otimes [Dy_i, \beta_i]$, where $p_i = [\alpha_i, x_i] \in R$ and $q_i = [y_i, \beta_i] \in L$. Then D_γ is a derivation on $R \hat{\otimes}_\gamma L$ and $\|D_\gamma\| \leq 2\|D\|$.

Proof. Clearly, $D_\gamma(u+v) = D_\gamma(u) + D_\gamma(v)$, $u = \sum_i p_i \otimes q_i \in R \hat{\otimes}_\gamma L$, $v = \sum_j r_j \otimes s_j \in R \hat{\otimes}_\gamma L$.

If $a \in F$, then

$D_\gamma(au) = D_\gamma(a \sum_i [\alpha_i, x_i] \otimes [y_i, \beta_i])$

$= D_\gamma(\sum_i [a\alpha_i, x_i] \otimes [y_i, \beta_i])$

$= \sum_i [a\alpha_i, Dx_i] \otimes [y_i, \beta_i] + \sum_i [a\alpha_i, x_i] \otimes [Dy_i, \beta_i]$

$= a(\sum_i [\alpha_i, Dx_i] \otimes [y_i, \beta_i] + \sum_i [\alpha_i, x_i] \otimes [Dy_i, \beta_i])$

$= a D_\gamma(u)$

Thus, D_γ is linear.

Now, $D_\gamma(uv) = D_\gamma((\sum_i p_i \otimes q_i)(\sum_j r_j \otimes s_j)) = D_\gamma((\sum_i [\alpha_i, x_i] \otimes [y_i, \beta_i])$.

$(\sum_j [\xi_j, z_j] \otimes [w_j, \delta_j]))$

$= D_\gamma(\sum_{i,j} [\alpha_i, x_i\xi_j z_j] \otimes [y_i\beta_i w_j, \delta_j])$

$= \sum_{i,j} [\alpha_i, D(x_i\xi_j z_j)] \otimes [y_i\beta_i w_j, \delta_j] + \sum_{i,j} [\alpha_i, x_i\xi_j z_j] \otimes [D(y_i\beta_i w_j)\ \delta_j]$

$= \sum_{i,j} [\alpha_i, D(x_i)\xi_j z_j + x_i\xi_j D(z_j)] \otimes [y_i\beta_i w_j, \delta_j] + \sum_{i,j} [\alpha_i, x_i\xi_j z_j] \otimes [D(y_i)\beta_i w_j + y_i\beta_i D(w_j)\ \delta_j]$

$= \sum_{i,j} [\alpha_i, D(x_i)\xi_j z_j] \otimes [y_i\beta_i w_j, \delta_j]$

$+ \sum_{i,j} [\alpha_i, x_i\xi_j D(z_j)] \otimes [y_i\beta_i w_j, \delta_j]$

$+ \sum_{i,j} [\alpha_i, x_i\xi_j z_j] \otimes [D(y_i)\beta_i w_j, \delta_j]$

$+ \sum_{i,j} [\alpha_i, x_i\xi_j z_j] \otimes [y_i\beta_i D(w_j)\ \delta_j]$

$= \sum_{i,j} ([\alpha_i, D(x_i)] \otimes [y_i, \beta_i])([\xi_j, z_j] \otimes [w_j, \delta_j]) + \sum_{i,j} ([\alpha_i, x_i] \otimes [y_i, \beta_i])([\xi_j, D(z_j)] \otimes [w_j, \delta_j])$

$+ \sum_{i,j} ([\alpha_i, x_i] \otimes [D(y_i)\ \beta_i])([\xi_j, z_j] \otimes [w_j, \delta_j]) + \sum_{i,j} ([\alpha_i, x_i] \otimes [y_i, \beta_i])([\xi_j, z_j] \otimes [D(w_j)\ \delta_j])$

$= \sum_i ([\alpha_i, D(x_i)] \otimes [y_i, \beta_i] + [\alpha_i, x_i] \otimes [D(y_i)\ \beta_i])$.

$(\sum_j [\xi_j, z_j] \otimes [w_j, \delta_j]) + (\sum_i [\alpha_i, x_i] \otimes [y_i, \beta_i]).\sum_j ([\xi_j, D(z_j)] \otimes [w_j, \delta_j] + [\xi_j, z_j] \otimes [D(w_j)\ \delta_j])$

$= D_\gamma(u)v + uD_\gamma(v)$

D_γ is a derivation on $R \hat{\otimes}_\gamma L$.

$\|D_\gamma(u)\|_\gamma = \|D_\gamma(\sum_i p_i \otimes q_i)\|_\gamma$

$\leq \|\sum_i [\alpha_i, D(x_i)] \otimes [y_i, \beta_i] + \sum_i [\alpha_i, x_i] \otimes [D(y_i)\ \beta_i]\|_\gamma$

$$\leq \sum_i \left(\|[\alpha_i, D(x_i)]\| \|[y_i, \beta_i]\| + \|[\alpha_i, x_i]\| \|[D(y_i)\ \beta_i]\| \right)$$

$$\leq \sum_i \left(\|\alpha_i\| \|D(x_i)\| \|y_i\| \|\beta_i\| + \|\alpha_i\| \|x_i\| \|D(y_i)\| \|\beta_i\| \right) \leq \left(\|D\| + \|D\| \right) \sum_i \|\alpha_i\| \|x_i\| \|y_i\| \|\beta_i\|$$

$$= 2\|D\| \sum_i \|[\alpha_i, x_i]\| \|[y_i, \beta_i]\| = 2\|D\| \sum_i \|p_i\| \|q_i\| \leq 2\|D\| (\|u\|_\gamma + \varepsilon)$$

Since ε is arbitrary, $\|D_\gamma(u)\|_\gamma \leq 2\|D\| \|u\|_\gamma$. So, $\|D_\gamma\| \leq 2\|D\|$.

It can be easily proved that the derivation D_γ defined above satisfies the following properties:

Theorem 2.9 (i) If D is a generalized derivation on M, then D_γ is also a generalized derivation.

(ii) Let M satisfies the condition $M\Gamma x=0 \Rightarrow x=0$ (or, $x\Gamma M=0 \Rightarrow x=0$).Then $2Sp(D) \subseteq Sp(D_\gamma)$.

(iii) If D is a Jordan derivation, then for $x=p \in R$, $D_\gamma(x^2)=xD_\gamma(x)+D_\gamma(x)x$.

In [9], Hejazian and Talebi introduced an ideal $M(D)$ of a derivation D on a Banach algebra A as:

$M(D)=\{x \in S(D) \cap J : D(x) \in J\}$, where $S(D)$ is the separating space of D and J is the Jacobson radical of A. It is proved that D is continuous if and only if $M(D)=\{0\}$.

Using Theorem 2.8, we have, if $M(D)=\{0\}$, then $M(D_\gamma)=\{0\}$. Here, we can raise the following problem:

Let $M(D_\gamma)=\{0\}$, for the derivation D_γ on R. From this can we derive some kind of continuity for the derivation D on M?

The concept of *derivative random operators* is presented in [12]. Let (Ω,β,P) be a complete probability measure space and (X, β_X) a measurable space, where X is a Banach space and β_X is the σ-algebra of all Borel subsets of X. A mapping $F: \Omega \to X$ is said to be X-valued random variable if the inverse image under F of every Borel set B belongs to β, i.e., $F^{-1}(B) \in \beta$ for all $B \in \beta_X$ (refer to [11]). A linear random operator F on a Γ-Banach algebra M is called *stochastically derivative* if

$P[F(x_1 \alpha x_2)=(Fx_1)\alpha x_2+x_1\alpha(Fx_2)]=1$ " $x_1, x_2 \in M$, $\alpha \in \Gamma$. F is called *probably derivative* if there exists $0<\delta<1$ such that $P[F(x_1\alpha x_2)=(Fx_1)\alpha x_2+x_1\alpha(Fx_2))] \geq \delta$, " $x_1, x_2 \in M$. Now, the following problem arises:

Given a stochastically derivative random operator F on the Γ-Banach algebra M, can we obtain a stochastically derivative random operator on R?

Similar problem can be raised in case of probably derivative random operator also.

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