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Thermoelastic Analysis on A Circular Plate Subjected to Annular Heat Supply

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ABSTRACT

Closed form expressions are developed for determining the quasi-static thermal stress in a circular plate with boundary conditions of radiation type subjected to annular heat flux, the curved surfaces being assumed insulated. It is assumed that temperature field is axisymmetric along the circumference of a circle over the upper face with zero temperature on the lower face. Numerical estimates for heating medium are obtained in terms of Bessel's functions and depicted graphically.

KEYWORDS : Quasi-static, transient response, circular plate, annular heat source, temperature distribution, thermal stress 2001 MSC: Primary 35A25; Secondary 74M99, 74K20

Introduction

The research on the prediction of stresses in circular plate with distributed heat supply has never ceased because of the importance of their basic structures in numerous civil, mechanical, electrical and computer engineering applications. However, it is evident from the list of existing literature that the stresses influenced by annular heat supply have not been studied theoretically despite of various recognized advantages. Earlier Nowacki (1957) has determined steady-state thermal stresses in a circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and circular edge. Roy Choudhuri (1972) has succeeded in determining the quasi-static thermal stresses in a circular plate subjected to transient temperature along the circumference of upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Wankhede (1982) has determined the quasi-static thermal stresses in circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature. Recently Noda et al. (1997) have considered a circular plate and discussed the transient thermoelastic-plastic bending problem, making use of the strain increment theorem. Khobragade et al. (2005) have studied the problem of partially distributed the heat supply of a thin circular disc applying finite Hankel and finite Fourier transform with Dirichlet type boundary conditions. In a recent work Varghese et al. (2006) have studied the inverse thermoelastic problem of a thin annular plate with third kind boundary conditions using integral transform techniques. In all aforementioned investigations an axisymmetrically heated plate has been considered. They have not however considered any particular distribution of temperature, which satisfies the time-dependent heat conduction equation and its stresses.

In the present paper an attempt is made to study the theoretical solution of a thermoelastic problem to determine the temperature distribution, stress functions and small deflection of a thin circular plate with boundary conditions of radiation type and subjected to known partially distributed annular heat supply occupying the space $D = \{(x, y, z) \in R^3 :$

$0 \le (x^2 + y^2)^{1/2} \le a, -h/2 \le z \le h/2\}$,

where $r = (x^2 + y^2)^{1/2}$. However, it is evident from the lists of existing literature that this particular problem influenced by annular heat supply on a thin circular plate has not been considered by any author despite of various recognized advantages.

Formulation of the problem:

Governing equation

In the first instance, we consider a circular plate of radius a and thickness h subjected to partially distributed heating of a concentric annular region of the upper face as shown in fig. 5. The plate is kept at zero initial temperature. The radiation flux is modeled as a 360-degree annular region occupying the space $D_1 : R_1 < r < R_2$ with internal and external radii designated as R_1 and R_2,

respectively. The plate is subjected to axisymmetric annular heat supply $Q_0 \hat{f}(r,t)/\lambda$. In order to simplify the analytical process, we assume the shape function $\hat{f}(r,t)$ for the surface heat generation having Q_0 as the heat flux with constant strength and λ as the thermal conductivity of the material. For convenience, we consider the undergiven functions as simpler function:

$$\hat{f}(r,t) = f(r)\,g(t), \quad f(r) = H(r-R_1) - H(r-R_2)$$

and hence, the annular region $D_1 : R_1 < r < R_2$ of the upper face is subjected to temperature distribution as follows:

$$T(r,z,t)\big|_{z=h/2} = -(Q_0/\lambda) f(r)\,g(t) \tag{1}$$

where

$$H(\hat{x}) = \begin{cases} 1, & \hat{x} \geq 0, \\ 0, & \hat{x} < 0; \end{cases}$$

The differential equation of the displacement function for the heating is given as

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{\partial^2 \phi}{\partial z^2} = a_t (1+\upsilon) T \tag{2}$$

with

$$\phi' = 0 \text{ at } r = a \text{ for all time } t \tag{3}$$

where the prime (‘) denotes differentiation with respect to r, $\phi(r,z,t)$ is the displacement component, υ and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the circular plate respectively and $T(r,z,t)$ is the heating temperature of the plate at time t satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{4}$$

where $\kappa = K/\rho c$ is the thermal diffusivity of the material of the plate, K is the conductivity of the medium, c is its specific heat and ρ is its calorific capacity, subject to the boundary conditions having medium $0 \leq r \leq a$ and $-h/2 \leq z \leq h/2$ as

$$M_r(T, \bar{k}_1, \bar{k}_2, a) = 0 \tag{5}$$

$$M_z(T, 0, 1, -h/2) = 0,$$

$$M_z(T, 0, 1, h/2) = -(Q_0/\lambda) f(r)\,g(t) \tag{6}$$

The most general expression for these conditions can be given by

$$M_{\vartheta}(\chi, \bar{k}, \bar{\bar{k}}, \$) = (\bar{k}\chi + \bar{\bar{k}}\chi')_{\vartheta=\$}$$

and

$$f(r) = \begin{cases} 1, & R_1 < r < R_2, \\ 0, & 0 < r < R_1,\ R_2 > r > a; \end{cases} \tag{7}$$

where the prime (′) denotes differentiation with respect to ϑ, radiation constants are $\bar{k}_1$ and $\bar{k}_2$ on the curved surfaces of the plate respectively.

For the particular solution we take stress components as

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial \phi}{\partial r} \tag{8}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 \phi}{\partial r^2} \tag{9}$$

$$\sigma_{rz} = \sigma_{r\theta} = \sigma_{\theta z} = \sigma_{zz} = 0,$$

if $T(r,z,t)$ and hence $\phi(r,z,t)$ are independent of θ. The constant μ is the modulus of rigidity. The deflection $\omega(r,t)$ of the circular plate satisfying the differential equation [2] as

$$D\nabla_1^4 \omega = -\frac{\nabla_1^2 M_T}{(1-\upsilon)} \tag{10}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{11}$$

and the resultant thermal momentum as

$$M_T(r,t) = a_t E \int_{-h/2}^{h/2} z\,T(r,z,t)\,dz \tag{12}$$

For assumption that the edge of the plate is fixed and clamped, we have

$$\omega = \frac{\partial \omega}{\partial r} = 0 \text{ at } r = a \tag{13}$$

Thus, the equations (4) to (13) constitute the mathematical formulation of the problem under consideration.

Solution of the heat conduction equation:

Determination of the temperatures $T(r,z,t)$

To obtain the expression for temperature $T(r,z,t)$, assume

$$T(r,z,t) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \sinh\{\gamma_n [z + (h/2)]\}\, g(t) \tag{14}$$

where α_1, α_2 ,…. are the positive roots of the equation $\alpha \bar{k}_1 J_0'(\alpha a) + \bar{k}_2 J_0(\alpha a) = 0$, $J_n(x)$ is the Bessel function of the first kind of order n and

$$J_n'(\alpha x) = \frac{d}{d(\alpha x)} J_n(\alpha x)$$

Substituting equation (14) in differential equation (4), one obtains

$$\gamma_n{}^2 = \alpha_n{}^2 + \frac{1}{\kappa}\frac{g'(t)}{g(t)}, n = 1,2 \qquad (15)$$

Also assume that

$$f(r) = \sum_{n=1}^{\infty} B_n J_0(\alpha_n r) \qquad (16)$$

Multiplying equation (16) by $r J_0(\alpha_n r)$ on both sides and integrating from 0 to a, one obtains from the theory of Bessel function

$$B_n \int_0^a r\left[J_0(\alpha_n r)\right]^2 dr = \int_0^a r f(r) J_0(\alpha_n r)\, dr = \int_{R_1}^{R_2} r\, J_0(\alpha_n r)\, dr$$

and using equation (7), we get

$$B_n = \frac{2\,[R_2 J_1(\alpha_n R_2) - R_1 J_1(\alpha_n R_1)]}{\alpha_n a^2[(J_0(\alpha_n a))^2 + (J_1(\alpha_n a))^2]} \qquad (17)$$

Substituting equation (14) into the second equation of equation (6) and using equations (5), one obtain

$$A_n = \frac{2Q_0\,\bar{k}_1\,[R_2 J_1(\alpha_n R_2) - R_1 J_1(\alpha_n R_1)]}{a^2\,\lambda (J_0(\alpha_n a))^2[\bar{k}_1\,\alpha_n + \bar{k}_2\,]\sinh\{\gamma_n h\}} \qquad (18)$$

Substituting equation (18) into equation (14), an axially symmetric temperature field is obtained as

$$T(r,z,t) = \frac{2Q_0\,\bar{k}_1}{a^2\,\lambda}\sum_{n=1}^{\infty} J_0(\alpha_n r)\left(\frac{R_2 J_1(\alpha_n R_2) - R_1 J_1(\alpha_n R_1)}{(J_0(\alpha_n a))^2[\bar{k}_1\,\alpha_n + \bar{k}_2\,]}\right)\left(\frac{\sinh\{\gamma_n[z + (h/2)]\}}{\sinh\{\gamma_n h\}}\right) g(t) \qquad (19)$$

Determination of displacement function

For obtaining the displacement function $\phi(r,z,t)$, we substitute temperature distribution equation (19) into equation (2) and using the result

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) J_0(\alpha_n r) = -\alpha_n^2\, J_0(\alpha_n r) \qquad (20)$$

one obtains

$$\phi(r,z,t) = -\frac{2Q_0\,(1+\nu)\,a_t\,\bar{k}_1}{a^2\lambda}\sum_{n=1}^{\infty}\frac{J_0(\alpha_n r)\Psi_n}{\alpha_n{}^2}\, g(t) \qquad (21)$$

where

$$\Psi_n = \left(\frac{R_2 J_1(\alpha_n R_2) - R_1 J_1(\alpha_n R_1)}{(J_0(\alpha_n a))^2[\bar{k}_1\,\alpha_n + \bar{k}_2\,]}\right)\left(\frac{\sinh\{\gamma_n[z + (h/2)]\}}{\sinh\{\gamma_n h\}}\right)$$

Determination of stress function

Substituting the equation (21) in equation (8) and (9), we get the required expression for thermal stresses as

$$\sigma_{rr} = -\frac{4\,\mu\,Q_0(1+\nu)a_t\,\bar{k}_1}{a^2\,\lambda}\frac{1}{r}\sum_{n=1}^{\infty}\frac{J_1(\alpha_n r)\Psi_n}{\alpha_n}\, g(t) \qquad (22)$$

$$\sigma_{\theta\theta} = -\frac{2\mu Q_0\,(1+\nu)a_t\,\bar{k}_1}{a^2\,\lambda}\sum_{n=1}^{\infty}[J_0(\alpha_n r) - J_2(\alpha_n r)]\,\Psi_n\, g(t) \qquad (23)$$

Determination of Quasi-Static small deflection

We assume that

$$\varpi(r,t) = \sum_{n=1}^{\infty} C_n(t)[J_0(\alpha_n r) - J_0(\alpha_n a)] \qquad (24)$$

Substituting equation (19) in equation (12) and using equation (20), one obtains

$$M_T(r,t) = \frac{Q_0\,a_t\,E\,\bar{k}_1}{a^2\,\lambda}\sum_{n=1}^{\infty} J_0(\alpha_n r)\,\Omega_n\, g(t) \qquad (25)$$

From equation (10), one obtains the $C_n(t)$ by substituting the equations (24) and (25) and using equation (20) as

$$C_n(t) = \frac{Q_0\,a_t\,E\,\bar{k}_1\;\alpha_n{}^2 J_0(\alpha_n r)\,\Omega_n\, g(t)}{D\,a^2\,\lambda(1-\upsilon)[\alpha_n{}^4 J_0(\alpha_n r) - J_0(\alpha_n a)]} \qquad (26)$$

Finally in order to obtain the required thermal deflection $\varpi(r,t)$, substitute the $C_n(t)$ from equation (26) into equation (24) we get

$$\varpi(r,t) = \frac{Q_0\,a_t\,E\,\bar{k}_1}{D\,a^2\,\lambda(1-\upsilon)}\sum_{n=1}^{\infty}\frac{\alpha_n^2 J_0(\alpha_n r)\,\Omega_n\,[J_0(\alpha_n r) - J_0(\alpha_n a)]}{\alpha_n^4 J_0(\alpha_n r) - J_0(\alpha_n a)}\, g(t) \qquad (27)$$

where

$$\Omega_n = \left(\frac{R_2 J_0(\alpha_n R_2) - R_1 J_0(\alpha_n R_1)}{(J_0(\alpha_n a))^2\gamma_n{}^2[\bar{k}_1\,\alpha_n + \bar{k}_2\,]}\right)\left(\frac{h\gamma_n(1+\cosh\{\gamma_n h\}) - 2\sinh\{\gamma_n h\}}{\sinh\{\gamma_n h\}}\right)$$

Special case

Set $g(t) = \exp(-\omega t)$ (28)

and

$$A = \frac{2Q_0\bar{k}_1}{a^2\lambda},\; B = -\frac{2Q_0(1+\nu)a_t\,\bar{k}_1}{a^2\lambda},\; C = \frac{Q_0\,a_t\,E\,\bar{k}_1}{D\,a^2\,\lambda(1-\upsilon)} \qquad (29)$$

Substituting the value of (28) in the equations (19), (21), (22), (23) and (27) one obtains

$$\frac{T(r,z,t)}{A} = \sum_{n=1}^{\infty} J_0(\alpha_n r)\Psi_n \exp(-\omega t) \qquad (30)$$

$$\frac{\phi(r,z,t)}{B} = \sum_{n=1}^{\infty}\frac{J_0(\alpha_n r)\Psi_n}{\alpha_n{}^2}\exp(-\omega t) \qquad (31)$$

$$\frac{\sigma_{rr}}{B} = 2\mu\frac{1}{r}\sum_{n=1}^{\infty}\frac{J_1(\alpha_n r)\Psi_n}{\alpha_n}\exp(-\omega t) \qquad (32)$$

$$\frac{\sigma_{\theta\theta}}{B} = \mu\sum_{n=1}^{\infty}[J_0(\alpha_n r) - J_2(\alpha_n r)]\,\Psi_n \exp(-\omega t) \qquad (33)$$

$$\frac{\varpi(r,t)}{C} = \sum_{n=1}^{\infty}\frac{\alpha_n{}^2 J_0(\alpha_n r)\,\Omega_n[J_0(\alpha_n r) - J_0(\alpha_n a)]}{\alpha_n{}^4 J_0(\alpha_n r) - J_0(\alpha_n a)}\exp(-\omega t) \qquad (34)$$

Numerical results, discussion and remarks

To interpret the numerical computation we consider material properties of low carbon

steel, which can be used for medium duty shafts, studs, pins, distributor cams, cam shafts, and universal joints having mechanical and thermal properties $\kappa = 13.97$ $\upsilon = 0.29$, $\lambda = 51.9$ and $a_t = 14.7$. With the general convention that the thickness of the thin circular plate is taken $\leq$ (diameter / 40) as $h = 0.1$, with radius $a = 1$ and radiation constant $\bar{k}_1 = \bar{k}_2 = 0.86$. $\alpha_n = 1.2558$, 4.0795, 7.1557, 10.2710, 13.3983, 16.5312, 19.6668, 22.8040, 25.9423, 29.0812, 32.2207, 35.3605, 38.5007, 41.6411, 44.7816, 47.9223, 51.0631, 54.2040, 57.3449, 60.4859 are the positive roots of the transcendental equation $\alpha \bar{k}_1 J_0'(\alpha) + \bar{k}_2 J_0(\alpha) = 0$ for $a = 1$. In the foregoing analysis will be illustrated by the numerical results shown in Figure 1 to 5. Figure 1 depicts the distributions of the temperature increment $T(r, z, t)$ verse radius at different values of time with $z = 0.1$. It shows that heat gain follows the sinusoidal nature crossing the inner core with increase of radius up to the outer region of radiation flux. But it was observed that while approaching towards the inner boundary region of radiation flux temperature increment increases slowly in the first quadrant having sinusoidal nature and then decrease slowly following the initial nature. The physical meaning emphasis for this phenomenon is that there is reduction in the rate of heat propagation before radius $r = 0.5$ up to $r = 0.75$ leading to compressive radial stress at inner part and expand more on outer due to partially distributed annular heat supply. Figure 2 and 3 shows the distributions of the radial and axial thermal stresses at different value of time. The radial stresses are smaller than axial stresses from figures 2-3. Figure 4 shows the variation of displacement function with same aforementioned parameter versus time. The corresponding radial displacement can be obtained using the theoretical formulae as

$$u_r = 2\alpha \frac{\partial \phi}{\partial z} - \nabla \phi$$

Conclusion

In this problem, we modify the conceptual idea proposed by Roy Choudhuri (1972) for circular plate and further investigated for the temperature distributions, displacement, stress function and deflection subjected to annular heat supply. The thermoelastic behaviour is examined with the help of arbitrary heat supply on the upper surface. We develop the analysis for the temperature field for heating processes by introducing the temperature function satisfying all boundary conditions of radiations type. Since the thickness of plate is very small, the series solution given here will be definitely convergent. Any particular case can be derived by assigning suitable values to the parameters and functions in the series expressions. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.

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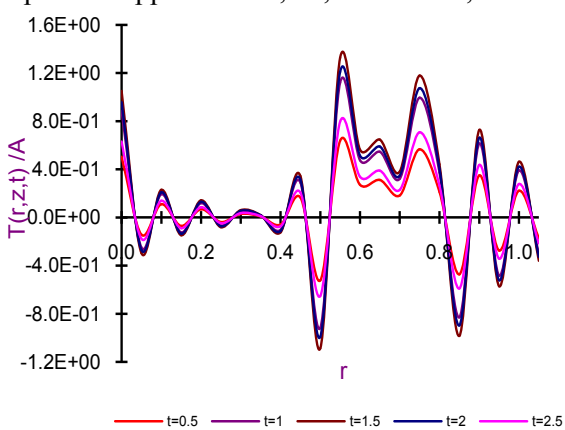


Figure (1)
Distribution of the temperature versus radius for different values of time

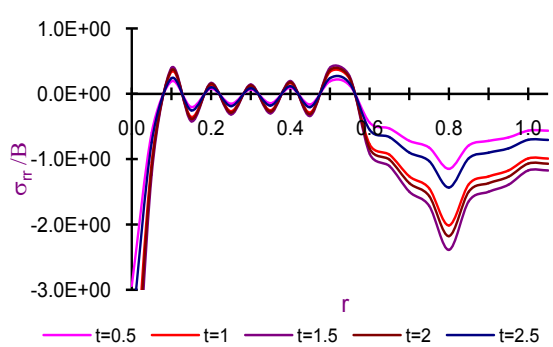


Figure (2)
Distribution of the radial stress versus radius for different values of time

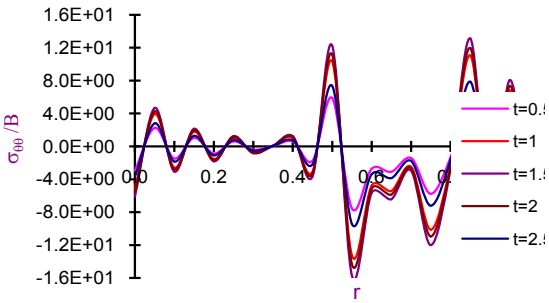


Figure (3)
Distribution of the tangential stress versus radius for different values of time

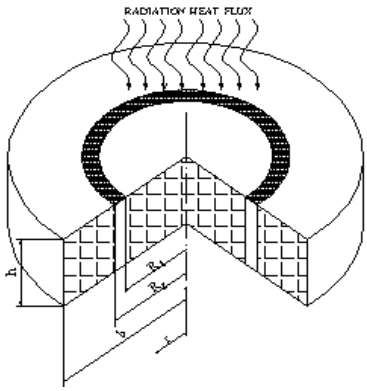


Figure (5)
Configuration of thin circular plate subjected to annular radiation flux

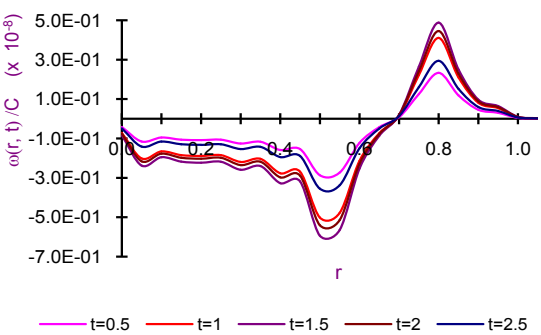


Figure (4)
Thermal deflection versus radius for different values of time

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