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Potential Energy Density Using Anisotropic Line Element

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ABSTRACT *Potential energy in the anisotropic coordinate system can be regarded as negligible as compared with Newtonian potential energy. On the other hand in the isotropic coordinate system both Potential Energy and Kinetic Energy have double the value of Newtonian Energy as will be proved later. But the sum of Potential Energy and Kinetic Energy is conserved just like Newton's theory and has got Newtonian approximation.*

KEYWORDS: Newtonian potential energy, Newtonian approximation

INTRODUCTION

The actual expression [1] (art.87, eqn.87.14, pp224) in conservative theorem is

$$\frac{\partial}{\partial x^{\nu}}\left[\overline{\Theta^{\nu}_{\mu}} + t^{\nu}_{\mu}\right] = 0$$

This is not a tensor equation but a covariant expression. This has been used to prove conservation of momentum and energy. One uses the theorem

$$\int_v \mathrm{div}V\,d\tau = \int_s V_n\,ds$$

to prove conservation of Energy and momentum. Now the fluid is an isolated material in empty space. Naturally the surface of the fluid divides the entire space into distinct regions namely exterior and interior. In the interior region fluid exits therefore ρ0 exists and e μ and e ν are of the type 1+α r+βr2.And for exterior region ρ_0 does not exist but e μ and e ν are of the type, $\frac{a}{r}+\frac{b}{r^2}$ etc. but the expression for $\left(\overline{\Theta^{\nu}_{\mu}} + t^{\nu}_{\mu}\right)$ covers both exterior and interior regions.

Now we are interested in the P.E. of the fluid so the terms in which 'r' occurs in the denominator will not be counted. Also any constant added to P. E. does not matter because it is the potential difference which matters. In the case of earth, each Christoffel symbol in the Cartesian system is of the order of 10-18, for earth, so the product of Christoffel symbols is of the order of 10-36, for earth. Since the expression for the total energy is covariant and the product of Cheristoffel symbols is of the order of 10-36 it is much less than ρ because ρ is of the order of 10-27 for earth.

We know that expression of total energy is covariant. Therefore for spherical coordinate system also the order of total energy is same. The field is weak, the space is almost flat. Now, at a point, the neighborhood can be regarded as a tangentially flat space. Hence it is possible to find out coordinates, so that $g_{\mu\nu} = \text{constant for } \mu = \nu$ and $g_{\mu\nu} = 0 \text{ for } \mu \neq \nu$ (In tangentially flat space Cheristoffel symbols vanish). Now if we calculate the total energy density in the tangentially flat space it will be $\rho\sqrt{-g}$.

Expression for t^4_4 and $\mathcal{L}$

Tolman gives [1] (art.87,eqn.87.12, pp224)

$$t^{\alpha}_{\beta} = \frac{1}{16\pi}\left[-\mathfrak{g}^{\mu\nu}_{\beta}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}_{\alpha}} + g^{\alpha}_{\beta}\mathcal{L} + 2g^{\alpha}_{\beta}\Lambda\sqrt{-g}\right]$$

$$\text{for } t^4_4\ ;\quad \alpha = 4,\ \beta = 4$$

$$\mathfrak{g}^{\mu\nu}_{\beta} = \frac{\partial\left(g^{\mu\nu}\sqrt{-g}\right)}{\partial\beta}$$

$$\mathfrak{g}^{\mu\nu}_{4} = \frac{\partial\left(g^{\mu\nu}\sqrt{-g}\right)}{\partial t}$$

$$\therefore \mathfrak{g}^{\mu\nu}_{4} = 0$$

Since we are considering time independent cases.

since for time independent case $\frac{\partial g^{\mu\nu}}{\partial t} = 0$

we are interested in t^4_4

$\therefore t^{\alpha}_{\beta} = \frac{1}{16\pi}\left[g^{\alpha}_{\beta}L\right]$, because $\beta = 4 = \alpha$ in the time independent case

as $\Lambda = 0$ for earth whose radius is much less than the solar system

$$t^4_4 = \frac{1}{16\pi}\left[g^4_4\mathcal{L}\right] = \frac{1}{16\pi}\left[\mathcal{L}\right]$$

$$g^4_{\mu} = 1 \text{ for } \mu = 4$$

$$g^4_{\mu} = 0 \text{ for } \mu \neq 4$$

Tolman [1] (page no.222, equ.87.1)

$$\mathcal{L} = \sqrt{-g}\,g^{\mu\nu}\left[\{\mu\alpha,\ \beta\}\{\nu\beta,\ \alpha\} - \{\mu\nu,\ \alpha\}\{\alpha\beta,\ \beta\}\right]$$

$\mathcal{L} = \sqrt{-g}$ (g11[{11,1}{11,1}+{11,2}{12,1}+{11,3}{13,1}+{11,4}{14,1}+{12,1}{11,2}+{12,2}{12,2}+{12,3}{13,2}+{12,4}{14,2}+{13,1}{11,3}+{13,2}{12,3}+{13,3}{13,3}+{13,4}{14,3}+{14,1}{11,4}+{14,2}{12,4}+{14,3}{13,4}+{14,4}{14,,4}]

-g11[{11,1}{11,1}+{11,1}{12,2}+{11,1}{13,3}+{11,1}{14,4}+{11,2}{21,1}+{11,2}{22,2}+{11,2}{23,3}+{11,2}{24,4}+{11,3}{31,1}+{11,3}{32,2}+{11,3}{33,3}+{13,3}{34,4}+{14,4}{41,1}+{11,4}{42,2}+{11,4}{43,3}+{11,4}{44,4}]

+g22[{21,1}{21,1}+{21,2}{22,1}+{21,3}{23,1}+{21,4}{24,1}+{22,1}{21,2}+{22,2}{22,2}+{22,3}{23,2}+{22,4}{24,2}+{23,1}{21,3}+{23,2}{22,3}+{23,3}{23,3}+{23,4}{24,3}+{24,1}{21,4}+{24,2}{22,4}+{24,3}{23,4}+{24,4}{24,4}]

-g22[{22,1}{11,1}+{22,1}{12,2}+{22,1}{13,3}+{22,1}{14,4}+{22,2}{21,1}+{22,2}{22,2}+{22,2}{23,3}+{22,2}{24,4}+{22,3}{31,1}+{22,3}{32,2}+{22,3}{33,3}+{22,3}{34,4}+{22,4}{41,1}+{22,4}{42,2}+{22,4}{43,3}+{22,4}{44,4}]

+g33[{31,1}{31,1}+{31,2}{32,1}+{31,3}{33,1}+{31,4}{34,1}+{32,1}{31,2}+{32,2}{32,2}+{32,3}{33,2}+{32,4}{34,2}+{33,1}{31,3}+{33,2}{32,3}+{33,3}{33,3}+{33,4}{34,3}+{34,1}{31,4}+{34,2}{32,4}+{34,3}{33,4}+{34,4}{34,4}]

-g33[{33,1}{11,1}+{33,1}{12,2}+{33,1}{13,3}+{33,1}{14,4}+{33,2}{21,1}+{33,2}{22,2}+{33,2}{23,3}+{33,2}{24,4}+{33,3}{31,1}+{33,3}{32,2}+{33,3}{33,3}+{33,3}{34,4}+{33,4}{41,1}+{33,4}{42,2}+{33,4}{43,3}+{33,4}{44,4}]

+g44[{41,1}{41,1}+{41,2}{42,1}+{41,3}{43,1}+{41,4}{44,1}+{42,1}{41,2}+{42,2}{42,2}+{42,3}{43,2}+{42,4}{44,2}+{43,1}{41,3}+{43,2}{42,3}+{43,3}{43,3}+{43,4}{44,3}+{44,1}{41,4}+{44,2}{42,4}+{44,3}{43,4}+{44,4}{44,4}]

-g44[{44,1}{11,1}+{44,1}{12,2}+{44,1}{13,3}+{44,1}{14,4}+{44,2}

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{21,1}+{44,2}{22,2}+{44,2}{23,3}+{44,2}{24,4}+{44,3}{31,1}+{44,3}{32,2}+{44,3}{33,3}+{44,3}{34,4}+{44,4}{41,1}+{44,4}{42,2}+{44,4}{43,3}+ {44, 4} {44,4}])

The anisotropic line element is

$$ds^2 = -e^{\lambda}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\varphi^2 + e^{\nu}dt^2.$$

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \quad g_{11} = -e^{\lambda} \quad g_{22} = -r^2 \quad g_{22} = -r^2\sin^2\theta \quad g_{44} = e^{\nu}$$

$$g^{11} = -e^{-\lambda} \quad g^{22} = -r^{-2} \quad g^{33} = -\frac{1}{r^2\sin^2\theta} \quad g^{44} = e^{-\nu}$$

Christoffel symbols corresponding to anisotropic line element are given

by Tolman [1](Page 241)

$\{11, 1\} = \frac{1}{2}\lambda' = A$ $\quad \{21, 2\} = \frac{1}{r} = B$

$\{12, 2\} = \frac{1}{r} = B$ $\quad \{22, 1\} = -re^{-\lambda} = D$

$\{13, 3\} = \frac{1}{r} = B$ $\quad \{23, 3\} = \text{Cot}\theta = E$

$\{14, 4\} = \frac{1}{2}\nu' = C$

$\{31, 3\} = \frac{1}{r} = B$ $\quad \{41, 4\} = \frac{1}{2}\nu' = C$

$\{32, 3\} = \text{Cot}\theta = E$ $\quad \{44, 1\} = \frac{1}{2}e^{\nu-\lambda}\nu' = H$

$\{33, 1\} = -r\text{Sin}^2\theta e^{-\lambda} = F$

$\{33, 2\} = -\text{Sin}\theta\,\text{Cos}\theta \quad = G$

Keeping those terms for which Christoffel symbols exist in the Lagrangian we get

$\mathcal{L} = \sqrt{-g}$ (g11[{12,2}{12,2}+{13,3}{13,3}+{14,4}{14,4}-{11,1}{12,2}-{11,1}{13,3}-{11,1}{14,4}]

+ g^{22}[{21,2}{22,1}+{22,1}{21,2}+{23,3}{23,3}- {22,1}{11,1}-{22,1}{12,2}-{22,1}{13,3}-{22,1}{14,4}]

+g^{33}[{31,3}{33,1}+{32,3}{33,2}+{33,1}{31,3}+{33,2}{32,3}-{33,1}{11,1}-{33,1}{12,2}-{33,1}{13,3}-{33,1}{14,4}- {33, 2} {23, 3} + {33, 2} {24, 4}]

+g^{44}[{41,4}{44,1}+{44,1}{41,4}- {44,1}{11,1}- {44,1}{12,2}-{44, 1} {13, 3}-{44, 1} {14, 4}])

$\mathcal{L} = \sqrt{-g}$ (g11 [B2+B2 +C2 –AB-AB-AC] +g22 [BD+BD +E2 –AD-BD-CD] +g33 [BF+EG+BF+EG-AF-BF-BF -CF-EG] +g44 [CH+CH-AH-BH-CH])

$\mathcal{L} = \sqrt{-g}$ (g11 [2 B2+ C2-2AB-AC]+g22[E2-AD-CD]+g33[EG-AF-CF] +g44 [CH-AH-2BH]) ... (1)

g^{11} [2 B^2+ C^2-2AB-AC]

$$= g^{11}\left[2\frac{1}{r^2} + \frac{\nu'^2}{4} - \frac{2\lambda'}{2}\frac{1}{r} - \frac{\lambda'\nu'}{4}\right]$$

$$= -e^{-\lambda}\left[\frac{\nu'^2}{4} + \frac{2}{r^2} - \frac{\lambda'}{r} - \frac{\lambda'\nu'}{4}\right]. \quad \text{... (a)}$$

$$g^{22}\left[E^2 - AD - CD\right]$$

$$= g^{22}\left[\text{Cot}^2\theta + \frac{\lambda'}{2}re^{-\lambda} + \frac{\nu'}{2}re^{-\lambda}\right]$$

$$= -\frac{1}{r^2}\left[\cot^2\theta + \frac{\lambda'}{2}re^{-\lambda} + \frac{\nu'}{2}re^{-\lambda}\right] \quad \text{.......(b)}$$

$$g^{33}\left[EG - AF - CF\right]$$

$$= g^{33}\left[-\cot\theta\sin\theta\cos\theta + \frac{\lambda'}{2}r\sin^2\theta e^{-\lambda} + \frac{\nu'}{2}r\sin^2\theta e^{-\lambda}\right]$$

$$= \frac{1}{r^2\sin^2\theta}\left[-\cos^2\theta + \frac{\lambda'}{2}r\sin^2\theta e^{-\lambda} + \frac{\nu'}{2}r\sin^2\theta e^{-\lambda}\right]$$

$$= \frac{1}{r^2}\left[-\cot^2\theta + \frac{\lambda'}{2}re^{-\lambda} + \frac{\nu'}{2}re^{-\lambda}\right] \quad \text{.........(c)}$$

$$g^{44}[CH - AH - 2BH] = g^{44}\left[\frac{\nu'}{2}\frac{\nu'}{2}e^{\nu-\lambda} - \frac{\lambda'}{2}\frac{\nu'}{2}e^{\nu-\lambda} - 2\frac{1}{r}\frac{\nu'}{2}e^{\nu-\lambda}\right]$$

$$= e^{-\nu}\left[\frac{\nu'^2}{4}e^{\nu-\lambda} - \frac{\lambda'\nu'}{4}e^{\nu-\lambda} - \frac{\nu'}{r}e^{\nu-\lambda}\right]$$

$$= e^{-\lambda}\left[\frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} - \frac{\nu'}{r}\right] \quad \text{...............(d)}$$

∴ Substitute equation (a), (b), (c) and (d) in Equation (1),

$$\mathcal{L} = \sqrt{-g}\left(-e^{-\lambda}\left[\frac{\nu'^2}{4} + \frac{2}{r^2} - \frac{\lambda'}{r} - \frac{\lambda'\nu'}{4}\right] - \frac{1}{r^2}\left[\cot^2\theta + \frac{\lambda'}{2}re^{-\lambda} + \frac{\nu' r}{2}re^{-\lambda}\right]\right.$$

$$\left. - \frac{1}{r^2}\left[-\cot^2\theta + \frac{\lambda'}{2}re^{-\lambda} + \frac{\nu'}{2}re^{-\lambda}\right] + e^{-\lambda}\left[\frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} - \frac{\nu'}{r}\right]\right)$$

$$\mathcal{L} = \sqrt{-g}\left(e^{-\lambda}\left[\frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} - \frac{\nu'}{r} - \frac{\nu'^2}{4} - \frac{2}{r^2} + \frac{\lambda'}{r} + \frac{\lambda'\nu'}{4}\right]\right.$$

$$\left. - \frac{1}{r^2}\left[\cot^2\theta + \frac{\lambda'}{2}re^{-\lambda} + \frac{\nu'}{2}re^{-\lambda} - \cot^2\theta + \frac{\lambda'}{2}re^{-\lambda} + \frac{\nu'}{2}re^{-\lambda}\right]\right)$$

$$\mathcal{L} = \sqrt{-g}\left(e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{\nu'}{r} - \frac{2}{r^2}\right] - \frac{1}{r^2}\left[\lambda' re^{-\lambda} + \nu' re^{-\lambda}\right]\right)$$

$$\mathcal{L} = \sqrt{-g}\left(e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{\nu'}{r} - \frac{2}{r^2}\right] - e^{-\lambda}\left[\frac{\lambda'}{r} + \frac{\nu'}{r}\right]\right)$$

$$\mathcal{L} = \sqrt{-g}\left(e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{\nu'}{r} - \frac{2}{r^2} - \frac{\lambda'}{r} - \frac{\nu'}{r}\right]\right)$$

$$\mathcal{L} = \sqrt{-g}\left(e^{-\lambda}\left[-\frac{2\nu'}{r} - \frac{2}{r^2}\right]\right)$$

$$\mathcal{L} = e^{\frac{(\lambda+\nu)}{2}}r^2\sin\theta\left(e^{-\lambda}\left[-\frac{2\nu'}{r} - \frac{2}{r^2}\right]\right) \quad \text{... (10.2)}$$

since $\sqrt{-g} = e^{\frac{(\lambda+\nu)}{2}}r^2\sin\theta$

We now show that the actual total energy is given by the first term $\rho\sqrt{-g}$. We were misled by Tolman's interpretation of t_4^4 as Potential Energy density [1] (page 24,eq.87.12). Actually t_4^4 vanishes in a tangentially flat space and total energy of sphere of the fluid of uniform density is $\rho\sqrt{-g}$. $\sqrt{-g}$ in the anisotropic case is 1 i.e. $e^{\frac{(\lambda+\nu)}{2}} \cong 1$ at r=a. L in equation (2) has two terms which are not negligible since they are of the order of 10-9 and -2. The second term which is approximately -2 is probably a constant this does not matter in potential energy. The first term is a significant term. However both the terms vanish in the tangentially flat space. Besides these considerations the order of magnitudes have to be found in the Cartesian system. The line element of the anisotropic coordinate system can be obtained from the isotropic coordinate system by the transformation $r = r'e^{\frac{\mu}{2}}$. This transformation gives the dθ2 and dφ2 terms correctly but dr2 term does not convert to $e^{\mu}dr'^2$, because x2+y2+z2=r2

∴ rdr=xdx+ydy+zdz,

giving dr2 as containing the off diagonal terms dxdy, dydz and dxdz, so that it is difficult to interpret these two terms. And we have to be satisfied by the fact that the Christoffel symbols and therefore L vanish completely in tangentially flat space.

This difficulty does not arise in the isotropic case so it is useful to find an expression for the total energy in the case of isotropic coordinate system. The next chapter deals with total energy in the isotropic coordinate system. The other argument is that the anisotropic metric can be converted into isotropic by transformation of coordinates and isotropic line element does not give significant terms in t_4^4.

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