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Perturbed Copulas and Their Tail Dependencies

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ABSTRACT

In this paper we extend our investigations of a special class of perturbations of copulas introduced in [3]. We show that this kind of perturbations does not change the value of tail dependence of the original copulas. Subsequently we apply the studied class of perturbed copulas to modelling relations between returns of an investments in a selected important REIT indexes.

KEYWORDS : copula, perturbation of copula, tail dependence, Real Estate Investment Trust (REIT) index, returns of REIT indexes

INTRODUCTION

Fitting of an appropriate copula to real data is one of major tasks in applications of copulas. For this purpose, a large buffer of potential copulas has been designed (mainly parametric families). Once we know approximately a copula C appropriate to model the observed data, we look for a minor perturbation of C which fits better thanC itself.

For this reason we will investigate the class of perturbation given (for any copula C) by

$$\mathrm{C}_{H_\alpha^C}(u,v) = \mathrm{C}(u,v) + \mathrm{H}_\alpha^C(u,v), \qquad (1)$$

where for each $\alpha \in [0,1]$

$$\mathrm{H}_\alpha^C(u,v) = \alpha\big(u - \mathrm{C}(u,v)\big)\big(v - \mathrm{C}(u,v)\big). \qquad (2)$$

We have shown that for this class of perturbations, the values of the coefficients of the tail dependence are the same as their values for the original copula C.

PERTURBATION OF BIVARIATE COPULA, TAILDEPENDENCE

We will consider bivariate copulas givenby (1), where C is a fixed copula and "H" _α^C : [0,1]^2→[0,1] is a continuous function. Function "H" _α^C is called *perturbation factor*and copulais C_(H_α^C) is called *perturbation* of C by means of "H" _α^C .

In [3], the following result has been obtained.

Theorem 1. *Let C: [0,1]^2→[0,1] be a copula and define by (2). Then given by (1) is a copula for each and any copula* C.

For a better specifications of the tail of a distributions, Joe [2] introduced the lower and upper tail dependence coefficients λ_L and λ_U.

Definition 1. Let X and Y are continuous random variables with distributions functions F_X and F_Y and with copula C, the *lower tail dependence coefficient* is defined by

$$\lambda_L = \lim_{u\to 0^+} \frac{\mathrm{C}(u,u)}{u} \qquad (3)$$

and the *upper tail dependence coefficient* by

$$\lambda_U = \lim_{u\to 1^-} \frac{1\text{-}2u+\mathrm{C}(u,u)}{1-u} = \qquad (4)$$
$$= \lim_{v\to 0^+} \frac{2v\text{-}1+\mathrm{C}(1-v,1-v)}{v}.$$

We follow the approach of Patton [5] and consider a so-called *survival copula* derived from a given copula C (corresponding to the couple (X, Y)) by

$$\hat{\mathrm{C}}(u,v) = u + v - 1 + \mathrm{C}(1-u,1-v) \qquad (5)$$

which is the copula related to the couple (-X, -Y).

It is obvious thattheupper and lower tail dependence for a mixture of two copulas C= α C1+ (1-α)C2 are given by

$$\lambda_U(C) = \alpha\lambda_U(C_1) + (1-\alpha)\lambda_U(C_2),$$
$$\lambda_L(C) = \alpha\lambda_L(C_1) + (1-\alpha)\lambda_L(C_2).$$

Moreover, for the survival copula corresponding to C we have

$$\lambda_U(C) = \lambda_L(\hat{C}) and \lambda_L(C) = \lambda_U(\hat{C}).$$

As it can be seen from Definition 1, the tail dependence coefficients are connected with the diagonal section of the bivariate copula C, which is defined by the function

$$\delta_C : [0,1] \to [0,1],\ \delta_C(u) = \mathrm{C}(u,u). \qquad (6)$$

Combining (6) with (5) we obtain

$$\delta_{\hat{C}}(u) = 2u - 1 + \delta_C(1-u), \qquad (7)$$
$$\delta_C(u) = 2u - 1 + \delta_{\hat{C}}(1-u).$$

From (3) and (4) we get

Theorem 2.*Let be a copula given by (1) and (2). Then*

$$\lambda_U\left(C_{H_\alpha^C}\right) = \lambda_U(C) \tag{9}$$

and

$$\lambda_L\left(C_{H_\alpha^C}\right) = \lambda_L(C). \tag{10}$$

Proof. $\lambda_L\left(C_{H_\alpha^C}\right) = \delta'_{C_{H_\alpha^C}}(0^+) = \delta'_C(0^+) + \alpha \frac{d[u-\delta_C(u)]^2}{du}|(0^+)$.

Wehave

$$0 \le u - \delta_C(u) \le u \text{ for} \in [0,1],$$

$$\text{thus} \frac{d[u-\delta_C(u)]^2}{du}|(0^+) = 0$$

and $\lambda_L\left(C_{H_\alpha^C}\right) == \delta'_C(0^+) = \lambda_L(C)$.

The upper tail dependence is given by

$$\lambda_U\left(C_{H_\alpha^C}\right) = 2 - \delta'_{C_{H_\alpha^C}}(1^-) =$$

$$= 2 - \left(\delta'_C(1^-) + \alpha \frac{d[u-\delta_C(u)]^2}{du}|(1^-)\right).$$

From (7) and for $u = 1 - v$ we obtain

$$u - \delta_C(u) = v - \delta_{\hat{C}}(v)$$

and thus

$$\frac{d[u-\delta_C(u)]^2}{du}\bigg|(1^-) = \frac{d[v-\delta_{\hat{C}}(v)]^2}{dv}\bigg|(0^+) = 0.$$

Therefore $\lambda_U\left(C_{H_\alpha^C}\right) = 2 - \delta'_C(1^-) = \lambda_U(C)$.

Remark. *We can conclude that the perturbations given by (2) do not change the values of tail dependence coefficients of the original copulas C. So we can consider these perturbations (in some sense) as minor even for values of the parameter that are not close to 0.*

APPLICATIONS TO REAL DATA MODELLING

We have investigated the relations between 4 selected countries'(USA, Australia, Japan and UK) daily returns of the REIT (Real Estate Investment Trust) indexes in different time periods, determined by the recent global financial markets crises (July 1, 2008 – April 30, 2009). We have performed filtering of the returns of all individual REIT indexes (in order to avoid a possible violation of the i.i.d. property) by ARMA-GARCH models (separately for the individual considered time sub-periods). We have applied the fitting by copulas to the residuals of ARMA-GARCH filters. We considered models from strict Archimedean copulas (Joe CJ, Frank CF, Clayton CCl and Gumbel CG) families and their mixtures with corresponding survival copula as well as their perturbation given by (1). For estimation of parameters for each type of models we have used the maximum pseudo-likelihood method. For selecting the optimal models we have applied the Kolmogorov-Smirnov-Anderson-Darling (for which we have used the abbreviation AD) test statistics defined e.g. in [1].

We observed strongly decreasing trends (between the subsequent considered time periods) for the values of the Kendall correlation coefficients as well as the values of tail dependence coefficients for the optimal copula models (for most considered couples of filtered returns of REIT indexes).

Weconcluded that for all considered six couples of (filtered) returns of REIT indexes in three time periods and for all six couples of considered models the best perturbed models have lower values of AD criterion than the best models in the corresponding non-perturbed classes. Moreover, for a great majority (16/18) of considered 18 couples of (filtered) returns of REIT indexes, the non-perturbed models corresponding to the optimal perturbed ones attain the minimal values of the AD criterion among all considered non-perturbed classes of models for the (filtered) returns of REIT indexes (for two remaining couples they narrowly exceed the minimum values). Moreover, for 17 of 18 considered couples of (filtered) returns of REIT indexes the coefficients of the optimal models do not considerably differ from the values of optimal models in the corresponding classes of non-perturbed models. The only exception to this phenomenon could be attributed to very flat shapes of the respective pseudo-likelihood functions around their minimum values.

CONCLUDING REMARKS

In the theoretical part of the paper, we derived an important result for the special type of perturbed copulas, where perturbations do not change the values of tail dependencies.

Despite the theoretical fact that investigated class of perturbations does not change the tail dependence coefficients of the considered copulas, their use yielded a significant improvement (with respect to the AD criterion).

An innovative approach to construction of copulas is investigated from the point of view of tail dependencies and applications to financial time series wereperformed in this study.

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REFERENCES

Berg, D. and Bakken, H. (2006),"Copula Goodness-of-fit Tests: A Comparative Study".Working paper, University of Oslo and Norwegian Computing center. | [2]Joe, H. (1997) "Multivariate model and dependence concept". Monographs on Statistics and Applied Proba-bility, CHAPMAN AND HALL, 73. | [3]Mesiar, R., Komorníková, M., and Komorník, J. "Perturbation of bivariate copula". Fuzzy Sets and System, Article in Press | [4]Nelsen, R., B. (2007), "An introduction to copulas".Springer Series in Statistics, SPRINGER-VERLAG, New York. | [5]Patton, A. (2006), "Modelling Asymmetric Exchange Rate Dependence." International Economic Review Vol. 47, No. 2, 527-556. |