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Modified Epidemic Model with Immigration and Non-monotonic Incident Rate under Treatment

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ABSTRACT

An epidemic model with non-monotonic incident rate under a limited resource for treatment is proposed and analyzed by Kar and Batabyal[7]. We have reinvestigated the model by considering that the treatment rate is proportional to the number of infective when it is below the capacity and is a constant when the number of infective is larger than the capacity. Existence and stability of the diseases free and endemic equilibrium are considered. Numerical results are also provided to justify the stability.

KEYWORDS : Epidemiology, Endemic equilibrium, Treatment, Basic reproductive number

1. Introduction:

Mathematical Models have become important tools in analyzing the spread and control of infectious diseases. In this process, rate of incidence plays a crucial role. The incidence is an epidemiological model is the rate at which susceptible become infectious. Bilinear and std. incidence rate have been frequently used in classical epidemic models Capasso and Serio[2] introduced a saturated incidence rate $g(I)S$ into epidemic models, where $g(I)$ tends to a saturation level when I gets large, i.e., $g(I) = KI/(1+\alpha I)$. Mena Lorca and Hethcote [11] also analyzed an SIRS model with the same saturation incidence. Several different incidence rates have been proposed by other researchers. Nonlinear incidence rate of the form bI^pS^q were investigated by Liu. et.al. [10]. A very general form of non-linear incidence rate was considered by Derrick and Driessche [3]. Ruan and Wang [12] studied an epidemic model with a specific non-linear incidence rate $kI^2S/(1+\alpha I^2)$ and presented a detailed qualitative and bifurcation analysis of the model. A more general incidence $\lambda I^pS/(1+\alpha I^q)$ was proposed by many other researchers [1,4,9, 11,13,&14]. Xiao and Ruan [15] proposed an epidemic model with non-monotonic incidence rate $\lambda IS/(1+\alpha I^2)$. Besides the rate and nature of incidence, treatment plays an important role to control the spread of diseases. This model is investigated and analyzed by Kar and Batabyal [7].Wang[13] proposed a treatment function

$$T(I) = rI, if\ 0 \le I \le I_0$$

$$K_1, if\ I > I_0 \qquad (1)$$

Where $K_1 = rI_0$ for some fixed value I_0.Kar and Batabyal [7] proposed a SIR model with non-monotonic incidence rate suggested by Xiao and Ruan [15] incorporating the above treatment function and non-monotonic incidence rate under a treatment function.

2. The Mathematical Model:

Following Kar and Batabyal [7], the proposed model is

$$\frac{dS}{dt} = a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R + \mu \quad (2)$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha I^2} - (d+m)I - T(I) \quad (3)$$

$$\frac{dR}{dt} = mI - (d+\beta)R + T(I) \quad (4)$$

Where $S(t), I(t), R(t)$ denote the number of susceptible, infective, recovered individuals, respectively. a is the recruitment rate of the population, d is the natural death rate of the population, λ is the proportionality constant, m is the natural recovery rate of the infective individuals, β is the rate at which recovered individuals lose immunity and return to susceptible class, μ the increase of at a constant rate and α is the parameter measures of the psychological or inhibitory effects and all other parameters have the same meaning as given in [9]. In this work take the treatment function $T(I)$ defined by

$$T(I) = rI, if\ 0 \le I \le I_0$$

$$K_1, if\ I > I_0$$

This means that the treatment rate is proportional to the infective when the number of infective is less or equal to some fixed value I_0 and the treatment is

constant when the number of infective crosses the fixed value I_0 . In the next section, the stability of the model taking two different cases of treatment function is discussed.

Case I: SIR model with $0 \leq I \leq I_0$

3. Equilibrium States and their Stability:

When $T(I) = rI$,the model reduces to

$$\frac{dS}{dt} = a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R + \mu \quad (5)$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha I^2} - (d+m+r)I \quad (6)$$

$$\frac{dR}{dt} = (m+r)I - (d+\beta)R \quad (7)$$

The System of equation (5)-(7) always has the disease free equilibrium $E_0\left(\frac{a+\mu}{d}, 0,0\right)$ for any set of parameter values. For the endemic equilibrium $E^*(S^*, I^*, R^*)$ is the solution of

$$a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R + \mu = 0$$

$$\frac{\lambda IS}{1+\alpha I^2} - (d+m+r)I = 0$$

$$(m+r)I - (d+\beta)R = 0$$

We define the basic reproductive number as follows

$$R_0 = \frac{\lambda(a+\mu)}{d(d+m+r)} \quad (8)$$

Lemma 3.1:

$S(t) + I(t) + R(t) = \frac{a+\mu}{d}$ is an invariant manifold of the system attracting the first octant.

Proof:

Let $N(t) = S(t) + I(t) + R(t)$ then

$\frac{dN}{dt} = (a+\mu) - d\,N(t)$

This imply $N(t) = A_1 e^{-dt} + \frac{(a+\mu)}{d}$

Simple mathematical calculation shows that $N(t)$ tends to $\frac{(a+\mu)}{d}$ as t tends to infinity.

We rescale the system by

$x = \frac{\lambda I}{d+\beta}$, $y = \frac{\lambda R}{d+\beta}$, $T = (d+\beta)t$

and obtain

$$\frac{dx}{dT} = \frac{x(K-x-y)}{1-vx^2} - ux \quad (12)$$

$$\frac{dy}{dt} = wx - y \quad (13)$$

Where

$k = \frac{(a+\mu)\lambda}{d(d+\beta)}$, $u = \frac{d+m+r}{d+\beta}$,

v = $\frac{\alpha(d+\beta)^2}{\lambda^2}$, $w = \frac{m+r}{d+\beta}$

Theorem 3.2:

(i) When the basic reproductive number $R_0 \leq 1$, there exist no positive equilibrium of the system (12)-(13) and in that case the only disease free equilibrium (0, 0) is a stable node.

(ii) When $R_0 > 1$, there exists a unique positive equilibrium of the system (12) - (13), and in that case (0, 0) is an unstable saddle point. Also the condition for which the unique positive equilibrium will be locally stable is $x^* < {}^{P_4}/_{P_3}$.

Global Stability: To investigate the global stability of the disease free equilibrium it is sufficient to show that (I(t), R(t)) (0, 0). From here, it is clear that $S(t) \to (a+\mu)/d$.

Theorem 3.3:

If $R_0 < 1$, then the disease free equilibrium $E_0\left(\frac{a+\mu}{d}, 0,0\right)$ of the system (10)-(11) is globally stable. But when $R_0 > 1$, system (10)-(11) have unique positive equilibrium and further when $x^* < P_4/P_3$ that unique positive equilibrium must be locally stable. Again since the system have no limit cycle in the positive quadrant, $E^*(x^*, y^*)$ must be globally stable under the condition $R_0 > 1$ and $x^* < P_4/P_3$.

Case II: SIR model with $I > I_0$

4. Equilibrium States and their Stability:

In this case the model reduces to:

$$\frac{dS}{dt} = a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R + \mu \quad (20)$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha I^2} - (d+m)I - K_1 \quad (21)$$

$$\frac{dR}{dt} = mI - (d+\beta)R + K_1 \quad (22)$$

Since $S + I + R = \frac{a+\mu}{d}$ is invariant manifold of the system (20)-(22), the model reduce to

$$\frac{dI}{dt} = \frac{\lambda I\left(\frac{a+\mu}{d} - I - R\right)}{1+\alpha I^2} - (d+m)I - K_1 \quad (23)$$

$$\frac{dR}{dt} = mI - (d+\beta)R + K_1 \quad (24)$$

Theorem 4.1:

When $K > u_1 + c$, the system (25)-(26) has two positive equilibrium $(\overline{x_1}, \overline{y_1})$ and $(\overline{x_2}, \overline{y_2})$, where $\overline{x_1}, \overline{x_2}$ are two positive solutions of the equation (27) under the parametric restriction given by (29), moreover when the conditions (32) and (34) are satisfied at some equilibrium point, that equilibrium point must be asymptotically stable.

Numerical Solution:

Case I: $0 \le I \le I_0$, we choose the parameters as follows:

$a = 3, d = 0.1, \alpha = 0.5, \lambda = 0.3, \mu = 0.1, m = 0.01, r = 0.2, \beta = 0.1$.

Here the basic reproductive number $R_0 = 30 > 1$. For the above choice of parameters we see that all the three components $S(t), I(t), R(t)$ approach to their steady state values as time goes to infinity, the disease becomes endemic.

Again if we take $= 15, d = 2.5, \lambda = 0.5, \alpha = 1, \mu = 0.3, \beta = 0.5, m = 10\ and\ r = 0.1$, the value of the basic reproductive number becomes 0.2428571<1.

By rescaling, the system (12) & (13) reduces to

$$\frac{dx}{dT} = \frac{x(46.5 - x - y)}{1 + 0.2222x^2} - 1.55x ,$$

$$\frac{dy}{dT} = 1.05 - y$$

Here $(u-k)<1$, and hence there exists unique positive equilibrium point (x^*, y^*) where

$x^* = 9.075827$ and $y^* = 8.643645$.

For the above choice of parameters $P_3 = 9.162602 > 0$, $P_4 = 196.5196$,

$P_4/P_3 = 21.44801$

and therefore the sufficient condition for local stability satisfied.

Case II: $I > I_0$, the parameters are:

$a = 2.8, d = 0.0453, \alpha = 2, \lambda = 0.4,$

$\mu = 0.1, m = 0.01, k_1 = 0.7, \beta = 0.13$

In this case $S + I + R = \frac{a+\mu}{d} = 64.01766$ is invariant manifold. The system reduces to

$$\frac{dI}{dt} = \frac{0.4I(64.01766 - I - R)}{1 + 2I^2} - (0.0553)I - 0.7$$

$$\frac{dR}{dt} = (0.01)I - (0.1753)R + 0.7$$

The rescaling system reduces to

$$\frac{dx}{dT} = \frac{x(143.32019 - x - y)}{1 + (0.3841263)x^2} - (0.3154592)x - 9.111591$$

$$\frac{dy}{dT} = (0.5704507)x - y + 9.111591$$

Here $(u-K)<0$ and hence there exists unique positive equilibrium point(x^*, y^*). For the choice of parameters

$P_3 = 392.8589 > 0, P_4 = 3651.5791,$

$P_4/P_3 = 9.2949$ and therefore the sufficient condition for local stability satisfied.

5. Conclusion:

In this paper we see that the basic reproductive number plays an important role to control the disease. When $R_0 \leq 1$, there exists no positive equilibrium, and in that case the disease free equilibrium is globally stable, that is the disease dies out. But when $R_0 > 1$, the unique endemic equilibrium is globally stable under some parametric condition. Also we see that the treatment rate plays a major role to control the disease.

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