Volume-4, Issue-8, August-2015 • ISSN No 2277 - 8160

Research Paper | Mathematics

Semi-Symmetric Non Metric Connection on an Almost Contact Metric Manifold

Shashi Prakash — Department of Mathematics Faculty of Science & Technology M. G. Kashi Vidyapeeth Varanasi U. P., INDIA

ABSTRACT *In 1924, Friedmann and Schouten have introduced the idea of Semi-Symmetric linear connection in a differentiable manifold. In 1970, Semi-Symmetric Connection were studied by K. Yano in a Ricmannian manifold. In 2008, S. K. Chaubey and R. H. Ojha defined a new Semi-Symmetric non metric and Quarter Symmetric metric connections. The purpose of the present paper is to study some properties of Semi-Symmetric non metric connections in an almost contact metric manifold, several useful Algebraic and geometrical properties have been studied.*

KEYWORDS : Semi Symmetric non metric connection, Almost contact metric manifold.

1. Introduction

Let V^n be an n-dimensional c^∞ manifold and let there exists in V^n a vector valued linear function F, a vector Field T and an 1-form A such that

$$F^2 X = -X + A(X)T \quad 1.1$$

$$\overline{X} \ def^n \ FX \quad 1.2$$

For any vector field X. Then V^n is called an almost contact manifold. From (1.1) the following relations hold

$$\overline{T} = 0 \quad 1.3$$

$$A(\overline{X}) = 0 \quad 1.4$$

$$A(T) = 1 \quad 1.5$$

In addition, if in $V_,^n$ there exists a metric tensor g satisfying

$$g(FX, FY) = g(X,Y) - A(X)A(Y) \quad 1.6$$

And

$$g(X.T) = A(X) \quad 1.7$$

Then V^n is called an almost contact metric manifold.

Let (V^n, g) be an n-dimensional Riemannian manifold of class c^∞ endowed with a Riemannian metric g and let D be the Levi-Civita connection on V^n.

Let $\widetilde{B}$ be an linear Connection defined on V^n. The Torsion tensor $S(X, Y)$ at $\widetilde{B}$ is given by (Kobayashi and Nomizu 1963).

$$S(X,Y) = \widetilde{B}_X Y - \widetilde{B}_Y X - [X,Y] \quad 1.8$$

Where X and Y are arbitrary vector fields if the torsion tensor S is of the form

$$S(X,Y) = A(Y)X - A(X)Y \quad 1.9$$

Where A is a 1 –form, then $\widetilde{B}$ is called Semi-Symmetric connection.

The connection $\widetilde{B}$ is called a non metric connection if

$$(\widetilde{B}_X g)\,(Y,Z) = 2A(Y)\,g(X,Z) + 2A(Z)\,g(X,Y) \qquad 1.10$$

It is known that (S. K. Chaubey and R. H. Ojha) for a Semi-Symmetric non metric connection

$$(\widetilde{B}_X Y) = D_X Y - A(Y)X - g(X,Y)\,T \qquad 1.11$$

where T is a vector field satisfying

$$g(X,T) = A\,(X) \qquad \text{for any vector field } X.$$

Sharfuddin and Hussain defined a Semi-Symmetric metric connection in an almost contact manifold by identifying 1-form A at (1.9) with the contact 1-form η. i.e.,

$$S(X,Y) = \eta(Y)\,X - \eta(X)Y$$

In 1995, Mileva Prvanovic studied a Semi Symmetric metric connection in a locally decomposable Riemannian space whose torsion tensor T satisfies the condition

$$(\widetilde{B}_X T)\,(Y,Z) = A(X)\,S(Y,Z) + A(FX)\,F(S(Y,Z)) \qquad 1.12$$

where A is a 1-form and F is a tensor field of type (1.1). In this paper we study a Semi-Symmetric non metric connection on an almost contact metric manifold satisfying (1.10) and

$$(\widetilde{B}_X T)\,(Y,Z) = A(X)\,S(Y,Z) + A(\phi X)\,\phi(S(Y,Z)) \qquad 1.13$$

Where ϕ is the tensor field of type (1.1) of the almost contact metric manifold.

2. Definition: Let $\widetilde{B}$ be a linear connection defined by

$$\widetilde{B}_X Y = D_X Y - A(Y)X - g(X,Y)T$$

where D is a Riemannian connection. Thus a relation between the curvature tensor K and $\overline{R}$ at the connections D and $\widetilde{B}$ respectively are given by

$$\overline{R}(X,Y,Z) = K(X,Y,Z) - \beta(X,Z)Y + \beta(Y,Z)X - g(Y,Z)(D_X T - \pi(X)T)$$
$$+ g(X,Z)\,(D_Y T - \pi(Y)T) \qquad 2.1$$

Where $$\overline{R}(X,Y,Z) = \widetilde{B}_X \widetilde{B}_Y Z - \widetilde{B}_Y \widetilde{B}_X Z - \widetilde{B}_{[X,\,y]} Z \qquad 2.2$$

$$K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,\,y]} Z \qquad 2.3$$

and $$\beta(Y,Z)\ def^{\,n}\ (D_Y \pi)\,(Z) + \pi(Y)\pi(Z) + g(Y,Z)\,\pi(T) \qquad 2.4$$

Where $$\pi(T) = 1$$

is the curvature tension with Respect to the Riemannian connection D.

The weyl conformal curvature tensor at the manifold is defined by

$$C(X,Y,Z) = R(X,Y,Z) + \alpha(Y,Z)X - \alpha(X,Z)Y - g(Y,Z)LX - g(X,Z)LY \qquad 2.5$$

Where $$\alpha(Y,Z) = g(LX,Z) = \frac{1}{n-2} \; T(Y,Z) + \frac{r}{2(n-1)\,(n-2)} g(Y,Z) \qquad 2.6$$

T and r denote respectively the (0, 2) Ricci tensor and Scalar curvature of the manifold.

The projective curvature tensor of the manifold is defined by

$$P(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}\left[S(Y,Z)X - S(X,Z)Y\right] \qquad 2.7$$

3. Curvature tensor of a Semi-Symmetric non metric connection:

We have to know that

$$S(Y,Z) = \eta(Z)Y - \eta(Y)Z \qquad 3.1$$

Where $$\eta(Z) = g(Z,\xi) \qquad 3.2$$

From (3.1) contracting Y we get

$$\left(E_1^1\ S\right)(Z) = (n-1)\ \eta(Z) \qquad 3.3$$

Now $$\left(\widetilde{B}_X E_1^1\ S\right)(Z) = (n-1)\ (\widetilde{B}_X\eta)(Z) \qquad 3.4$$

Let $$(\widetilde{B}_X\ S)(Y,Z) = A(X)\,S(Y,Z) + A(\phi X)\ \phi(S(Y,Z)) \qquad 3.5$$

Where A is a 1-form and ϕ is a tensor field of typed (1.1).

Contracting Y the above equation we get

$$(\widetilde{B}_X E_1^1\ S)(Z) = (n-1)\,A(X)\eta(Z) + a\ A(\phi X)\eta(Z) \qquad 3.6$$

Where $$a = \left(E_1^1\,\phi\right)Y \qquad 3.7$$

From (3.4) and (3.6)

$$(\widetilde{B}_X\eta)\ (Z) = A(X)\eta(Z) + bA(\phi X)\eta(Z) \qquad 3.8$$

Where $$b = \frac{a}{n-1} \qquad 3.9$$

Using (1.5) we get

$$(\widetilde{B}_X\eta)\ (Z) = (D_X\eta)(Z) - \eta(x)\eta(Z) - g(X,Z) \qquad 3.10$$

Using (3.8) and (3.10) we get

$$(D_X\eta)\ (Z) = A(X)\eta(Z) + bA(\phi X)\eta(Z) + \eta(X)\eta(Z) + g(X,Z) \qquad 3.11$$

From (2.4) and (3.11) we get

$$\beta(X,Z) = A(X)\eta(Z) + bA(\phi X)\eta(Z) + 2\{n(X)\eta(Z) + g(X,Z)\} \qquad 3.12$$

Therefore the curvature tensor $\overline{R}$ of the manifold with respect to the Semi-Symmetric non metric connection $\widetilde{B}$ is given by

$$\overline{R}(X,Y,Z) = K(X,Y,Z) + g(Y,Z)\{2X - \overline{X} + \pi(X)T\} + g(X,Z)\{\overline{Y} - 2Y - \pi(Y)T\}$$

$$-\eta(Z)Y\{A(X) + bA(\phi X) + 2\eta(X)\} + \eta(Z)X\{A(Y) + bA(\phi Y) + 2\eta(Y)\} \qquad 3.13$$

Where K denotes the curvature tensor of the manifold.

Theorem- 3.1: The curvature tensor with respect to $\widetilde{B}$ of an almost contact metric manifold admitting a Semi-Symmetric non metric connection $\widetilde{B}$ is

$$'K(X,Y,Z) = 2\eta(Z)\{\eta(X)g(Y,V) + +\eta(Y)g(X,V)\} - A(\xi)\eta(Z)$$

$$\{\eta(Y)g(X,V) - \eta(X)g(Y,V)\} - g(Y,Z)\{2\eta(X,V) - g(\overline{X},V)\} - g(X,Z)$$

$$\{g(\overline{Y},V) - 2g(Y,V)\} - \eta(V)\{g(Y,Z)\pi(X) - g(X,Z)\pi(Y)\} \qquad 3.14$$

Proof: From (3.13) it is obvious that

$$\overline{R}(X,Y,Z) = -\overline{R}(Y,X,Z) \qquad 3.15$$

We now define a tensor $'\overline{R}$ of the type (0, 4) by

$$'\overline{R}(X,Y,Z,V) = g(\overline{R}(X,Y,Z),V) \qquad 3.16$$

Equation (3.13) and (3.16) we obtain that

$$'\overline{R}(X,Y,Z,V) = -'\overline{R}(Y,X,Z,V) \qquad 3.17$$

Again from (3.13) we get

$$\overline{R}(X,Y,Z) + \overline{R}(Y,Z,X) + \overline{R}(Z,X,Y) = \{A(X) + bA(\phi X) + 2\eta(X)\}$$

$$\{\eta(Y)Z - \eta(Z)Y\} + \{A(Y) + bA(\phi Y) + 2\eta(Y)\}\{\eta(Z)X - \eta(X)Z\}$$

$$+\{A(Z) + bA(\phi Z) + 2\eta(Z)\}\{\eta(X)Y - \eta(Y)X\} \qquad 3.18$$

This is the first Bianchi identity with respect to $\widetilde{B}$ let $\overline{T}$ and T denote respectively the Ricci tensor of the manifold with respect to $\widetilde{B}$ and D

From 3.13 contracting X

$$\overline{T}(Y,Z) = T(Y,Z) + 3(n-1)g(Y,Z) + g(\overline{Y},Z) + (n-1)A(Y)\eta(Z)$$

$$+2(n-1)\eta(Y)\eta(Z) + (n-1)bA\phi(Y)\eta(Z) \qquad 3.19$$

$$A(Y) + bA(\phi Y) = A(\eta)\xi(Y)$$

Using the result from (3.18) we get

$$\overline{R}(X,Y,Z) + \overline{R}(Y,Z,X) + \overline{R}(Z,X,Y) = A(\xi)\{\eta(X) + \eta(Y) + \eta(Z)\} \qquad 3.26$$

Conversely we assume that (3.26) holds, then in virtue of (3.18) we have

$$\{A(X) + bA(\phi X) + 2\eta(X)\}\{\eta(Y)Z - \eta(Z)Y\} + \{(A(Y) + bA(\phi Y) + 2\eta(Y)\}$$

$$\{\eta(Z)X - \eta(X)Z\} + \{A(Z) + bA(\phi Z) + 2\eta(Z)\}\{\eta(X)Y - \eta(Y)X\}$$

$$= A(\xi)\{\eta(X) + \eta(Y) + \eta(Z)\} \qquad 3.27$$

Contracting X above equation we get

$$A(Z)\eta(Y) + bA(\phi Z)\eta(Y) - A(Y) - bA(\phi Y) = A(Y)\eta(Z) + bA(\phi Y)\eta(Z)$$

$$- A(Z) - bA(\phi Z)$$

Hence by (3.21) $\overline{T}$ is symmetric. **Proved**

ACKNOWLEDGEMENTS : First author is highly thankful to M. G. Kashi Vidyapith Varanasi-221002, for providing financial support to carry out the present research paper.

REFERENCES

1. K. YANO: On Semi-Symmetric metric connection. Revue Roumaine De Math. Pures at Appliques 15 (1970), 1579-1586 | 2. R. S. MISHRA, Structures on a differentiable manifold and their applications. Chandrama Prakashan, Allahabad (India), (1984). | 3. SHASHI PRAKASH AND S. D. SINGH, On an almost unified para-norden contact manifold. Ultra Scientist of Physical Sciences, Vol. 22(1)M, 27-30, (2010). | 4. R. H. OJHA AND S. PRASAD, On semi-symmetric metric S-connexion in a Sasakian manifold, Indian J. Pure and Appl. Math., Vol 16(4), 341-344, (1985). | 5. KOBAYASI S. AND NOMIZU K., Foundation of Differential Geometry, Vol. I, Reprint of the 1963 Original. Willely Classical Library, John Wiley and Sons, Inc., New York, (1996). | 6. YANO K. AND AKO M., Almost analytic vectors in almost complex spaces, Tohoku Math, J. Vol. 13, 24-45, (1961). | 7. SASAKI S. AND HATAKEYAMA Y., On differentiable manifolds with contact metric structures, J. Math Soc. Japan Vol. 14, 249-271, (1962). | 8. BOOTHBY, W. M., An Introduction to differentiable Manifolds and Riemannian Geometry, Academic Press, (1975). R. H. OJHA AND S. K. CHAUBEY, On a semi-symmetric non metric and quarter symmetric metric connections, Tensor N.S., vol. 70, No.2, 202-213, (2008). | 9. SHASHI PRAKASH AND S. D. SINGH, Tensors of the type (0,4) on an almost unified para-norden contact metric manifold. International Journal of mathematical analysis, Hikari Ltd., Vol.4, No. 14, 657-670, (2010). | 10. A. SHARFUDDIN and S.I. HUSSAIN: Semi-Symmetric metric Connections in Almost contact manifolds, Tensor, N. S. 30 (1976), 133-139.