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Numerical solutions of instabilities in displacement through porous media Using differential transform method

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ABSTRACT

In this paper, we present an analytical solution for instabilities in displacement through porous media's differential equations by using the differential transformation method. The convergence of this method has been discussed with example which is presented to show the ability of the method for linear and non-linear systems of differential equations.

KEYWORDS : Differential transform method, Reduced differential transform method

Introduction

Fluid flow instability is one of the important and classic problems of fluid Mechanics. The transition from laminar flow to turbulent flow and instability of the most common problems of fluid flow instability in recent years. Investigations of the instability of miscible or immiscible displacement has several applications in the industries with regards to the fluid flow in porous media. Therefore, the subject has been taken seriously by the researchers since the 1950s. This Phenomenon was modeled for the first time by Hill, in 1952[1].

The unsteady and unsaturated flow of water through soils is due to content changes as a function of time and entire pore spaces are not completely filled with flowing liquid respectively knowledge concerning such flows some helps some workers like hydrologist, agriculturalists, many fields of science and engineering. The water infiltrations water are encountered by these flows. Which are described by nonlinear partial differential equation.

The mathematical model conforms to the hydrological solution of one dimensional vertical ground water recharge by spreading. Such flow is of great importance in water resource science. Soil engineering and agricultural sciences.

If a fluid contained in a porous medium is displaced by another fluid of lesser viscosity, then it is frequently observed that the displacing fluid has a strong tendency to protrude in form of fingers (instabilities) into more viscous fluid. This phenomenon is called fingering.

In petroleum engineering, the fingering process is a well known phenomenon occurring in displacement of oil by water by flooding that is a common oil recovery technique.

Scheidegger and Johnson [2] discussed the statistical behavior in homogeneous porous media without capillary pressure. Verma [3] has examined the behavior of fingering in a displacement process through heterogeneous porous media. Bhathawala [4] discussed the analytical expression of the fingering in a displacement process through homogeneous porous media with mean capillary pressure.

Analysis of the numerical method

The basic definitions of RDTM are given below

If the function $u(x,t)$ is analytic and differential continuously with respect to time t and space x in the domain of interest, then let

$$U_k = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \qquad (1)$$

Where the t-dimensional spectrum function $U_k(x)$ is the transformed function $u(x,t)$ represent transformed function. The differential inverse transform of $U_k(x)$ is defined as follows

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k \qquad (2)$$

Then combining equation (1) & (2) we write

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k \qquad (3)$$

From the above definition, it can be found that the concept of RDTM is derived from the power series expansion.

Standard operator form of RDTM

$$L\big(u(x,t)\big) + R\big(u(x,t)\big) + N\big(u(x,t)\big) = g(x,t) \qquad (4)$$

with initial condition is

$$u(x,0) = f(x) \qquad (5)$$

Where $L\big(u(x,t)\big) = u_t(x,t)$is a linear operator which has partial derivatives.

$N(u(x,t))$ is a nonlinear term & $R(u(x,t))$ is the remaining linear term.

we can derive the following recursive relation.

$(k+1)U_{k+1}(x) = G_k(x) - N(U_k(x)) - R(U_k(x))$ (6)

Where $G_k(x)$, $N(U_k(x))$, $R(U_k(x))$ are transformations of the functions $g(x,t), N\big(u(x,t)\big), R(u(x,t))$ respectively.

We can write equation (5) as $u_0(x) = f(x)$ (7)

To find all other iterations, we first substitute equation (7) into equation (6) and then we find the values of $U_k(x)$ finally. We apply the inverse transformation to all values $\{U_k(x)\}_{k=0}^{n}$ to obtain the approximate solution.

$$\bar{u}_n(x,t) = \sum_{k=0}^{n} U_k(x) t^k \qquad (8)$$

Where n is the number of iterations we need to find the intended approximate solution.

Hence, the exact solution of our problem is given by

$$u(x,t) = \lim_{n\to\infty} \bar{u}_n(x,t) \qquad (9)$$

There are many physical problems can be described by mathematical models that involve linear & non-linear PDEs. Many authors applied numerical Methods to find solution of these equations and to name few of these methods. The differential transform method (DTM)[5, 6], The Adomian decomposition Method [7, 8] . The RDTM was first introduced by Y. Keskin in his Ph.D [9]. This method based on the numbers of iteration needed of the series solution for numerical purpose with high accuracy. In Recently RDTM [10, 11] derived analytic approximate solution to the Sharma Tasso Olver (STO) equation & exact solution to both Schrödinger equation and the Telegraph equation & solved gas dynamics equation.

Fingering in a homogeneous medium

Statement of the Problem

We consider that there is a uniform water injection into an oil saturated porous medium of homogeneous physical characteristics, such that the injecting water cuts through the oil formation and give rise to protuberance. This furnishes a well developed fingers flow. Since the entire oil at the initial boundary (x=0) is displaced through a small distance due to the water injection. Therefore, we assume, further that complete water saturation exists at the initial boundary.

Formulation of the Problem

The seepage of water (v_w) and oil (v_o) are given by Darcy's law,

$$v_w = -\left(\frac{k_w}{\delta_w}\right) K \left[\frac{\partial P_w}{\partial x}\right] \qquad (1)$$

$$v_o = -\left(\frac{k_o}{\delta_o}\right) K \left[\frac{\partial P_o}{\partial x}\right] \qquad (2)$$

Where K is the permeability of the homogeneous medium, k_w and k_o are relative permeability of water and oil, which are functions of s_w and s_o (s_w and s_o are the saturation of water and oil) respectively P_w and P_o are pressure of water and oil, δ_w and δ_o are constant kinematic viscosities, α is inclination of the bed and g is acceleration due to gravity.

Regarding the phase densities are constant, the equations of continuity of the two phases are:

$$P\left(\frac{\partial s_w}{\partial t}\right) + \frac{\partial v_w}{\partial x} = 0 \qquad (3)$$

$$P\left(\frac{\partial s_o}{\partial t}\right) + \frac{\partial v_o}{\partial x} = 0 \qquad (4)$$

Where P is porosity of the medium. From the definition of phase saturation, it is evident that,

$$s_w + s_o = 1 \qquad (5)$$

The capillary pressure P_c is defined as

$$P_c = -\beta_0 S_w \qquad (6)$$
$$P_c = P_o - P_w \qquad (7)$$

Where β_0 is a constant quantity.

At this state, for definiteness of mathematical analysis, we assume standard relationship due to Scheidegger and Jhonson [12] between phase saturation and relative permeability as

$$k_w = s_w \qquad (8)$$

$$k_o = s_o = 1 - s_w \qquad (9)$$

The equation of motion for saturation can be obtained by substituting the values of v_w and v_o from equation (1) and (2) into the equation (3) and (4) respectively, we get,

$$P\left(\frac{\partial s_w}{\partial t}\right) = \frac{\partial}{\partial x}\left[\left(\frac{k_w}{\delta_w}\right) K \left(\frac{\partial P_w}{\partial x}\right)\right] \qquad (10)$$

$$P\left(\frac{\partial s_o}{\partial t}\right) = \frac{\partial}{\partial x}\left[\left(\frac{k_o}{\delta_o}\right) K \left(\frac{\partial P_o}{\partial x}\right)\right] \qquad (11)$$

These are the general flow equations of the phase in homogeneous medium, when effects due to pressure discontinuity and gravity term in inclined porous medium are considered.

Eliminating $\left(\frac{\partial P_w}{\partial x}\right)$ from equation (10)

Now from equation (7) we obtain

$$P\left(\frac{\partial s_w}{\partial t}\right) = \frac{\partial}{\partial x}\left[\left(\frac{k_w}{\delta_w}\right) K \left(\frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x}\right)\right] \qquad (12)$$

Combining equation (11) and (12)

$$\frac{\partial}{\partial x}\left[K\left(\frac{k_w}{\delta_w} + \frac{k_o}{\delta_o}\right)\frac{\partial P_o}{\partial x} - K\left(\frac{k_w}{\delta_w}\frac{\partial P_c}{\partial x}\right)\right] = 0 \qquad (13)$$

Integrating above equation with respect to x, we have

$$K\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right)\frac{\partial P_o}{\partial x}-K\left(\frac{k_w}{\delta_w}\frac{\partial P_c}{\partial x}\right)=-V \qquad (14)$$

Where V is constant of integrating which can be evaluated from later on. Simplification of (14) gives

$$\frac{\partial P_o}{\partial x}=\frac{-V}{K\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right)}+\frac{\frac{\partial P_c}{\partial x}}{1+\left(\frac{k_o}{k_w}\right)\left(\frac{\delta_w}{\delta_o}\right)} \qquad (15)$$

Using above equation in (12), we have

$$P\left(\frac{\partial s_w}{\partial t}\right)+\frac{\partial}{\partial x}\left[\frac{-V}{K\left(\frac{k_w}{\delta_w}+\frac{k_o}{\delta_o}\right)}+\frac{\frac{\partial P_c}{\partial x}}{1+\left(\frac{k_o}{k_w}\right)\left(\frac{\delta_w}{\delta_o}\right)}\right]=0 \qquad (16)$$

The value of pressure of oil(P_o) can be written as [13] of the form

$$P_o=\frac{P_o+P_w}{2}+\frac{P_o-P_w}{2}=\bar{P}+\frac{1}{2}P_c \qquad (17)$$

Where $\bar{P}$ is the mean pressure which is constant, therefore (17) implies

$$\frac{\partial P_o}{\partial x}=\frac{1}{2}\frac{\partial P_c}{\partial x} \qquad (18)$$

Using above equation in (14), we get

$$V=\frac{K}{2}\left[\left(\frac{k_w}{\delta_w}\right)-\left(\frac{k_o}{\delta_o}\right)\frac{\partial P_c}{\partial x}\right] \qquad (19)$$

Substituting the value of V from above equation in equation (16), we get

$$P\left(\frac{\partial s_w}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left[K\left(\frac{k_w}{\delta_w}\right)\left(\frac{dP_c}{ds_w}\right)\left(\frac{\partial s_w}{\partial x}\right)\right] \qquad (20)$$

Substituting the value of k_w and P_c from (6) and (8) we get

$$P\left(\frac{\partial s_w}{\partial t}\right)-\frac{\beta_0}{2}\frac{K}{s_w}\frac{\partial}{\partial x}\left[s_w\frac{\partial s_w}{\partial x}\right]=0 \qquad (21)$$

A set of suitable boundary conditions associated to problem (21) are

$$s_w(0,t)=1;\ s_w(x,0)=0;\ s_w(L,t)=0 \qquad (22)$$

Equation (21) is reduced to dimensionless from by setting

$$X=\frac{x}{L},\ T=\frac{K\beta_0 t}{2\delta_w L^2 P} \qquad (23)$$

So that

$$\frac{\partial s_w}{\partial t}=\frac{\partial}{\partial x}\left(s_w\frac{\partial s_w}{\partial x}\right) \qquad (24)$$

Equation (24) is desired nonlinear differential equation of motion for the flow of immiscible liquid in homogeneous medium.

The problem is solved by using reduced differential transform method.

Solution of the Problem using reduced differential transform method:

$\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}(S_w \frac{\partial S_w}{\partial x})$

Taking the initial condition $S_w(x,0) = S_{w0} = f(x)$

This equation is nonlinear differential equation of motion for the flow of immiscible liquid in homogeneous medium.

The problem is solved by using Reduced differential transform method because our equation is partial differential equation.

- Reduced differential Transform Method
  The Basic definition of RDTM are given below
  If the function $u(x,t)$ is analytic and differential continuously with respect to time t and space x in the domain of interest then let

$$U_k = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]$$

Where the t-dimensional spectrum function $U_k(x)$ is the transformed function. $u(x,t)$ represent transformed function. The differential inverse transform of $U_k(x)$ is defined as follow

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k$$

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]t^k$$

Apply RDTM on (1)

$$(k+1)S_{w(k+1)}(x) = \frac{d}{dx}(s_{wk}(x))^2 + \left(s_{wk}(x)\frac{d^2}{dx^2}s_{wk}(x)\right)$$

$$S_{w0}(x) = f(x) = \frac{e^x - 1}{e - 1}$$

Put initial condition (3) into eq. (1), So we have the values of $U_k(x)$ as following.

$$S_{w1}(x) = \left(\frac{d}{dx}(s_{wk}(x)\right)^2 + \sum_{r=0}^{k}\left(s_{wr}(x)\frac{d^2}{dx^2}s_{w(k-r)}(x)\right)$$

$$S_{w1}(x) = \left(\frac{d}{dx}\left(\frac{e^x - 1}{e - 1}\right)\right)^2 + \left(\left(\frac{e^x - 1}{e - 1}\right)\frac{d^2}{dx^2}\left(\frac{e^x - 1}{e - 1}\right)\right)$$

$$S_{w1}(x) = \left(\frac{2e^{2x} - e^x}{(e-1)^2}\right)$$

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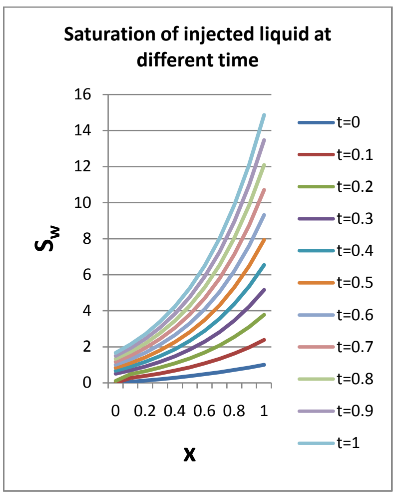
$$(1+1)S_{w2}(x) = \left(\frac{d}{dx}(s_{w1}(x))\right)^2 + \left[\left(s_{w0}(x)\frac{d^2}{dx^2}s_{w1}(x)\right) + \left(s_{w1}(x)\frac{d^2}{dx^2}s_{w0}(x)\right)\right]$$

$$(1+1)S_{w2}(x) = \left(\frac{d}{dx}\left(\frac{2e^{2x}-e^x}{(e-1)^2}\right)\right)^2 + \left(\left(\frac{2e^{3x}-e^{2x}}{(e-1)^3}\right) + \left(\frac{8e^{3x}-e^{2x}-8e^{2x}-e^x}{(e-1)^3}\right)\right)$$



$$S_{w2}(x) = \frac{1}{2}\left[\left[\left(\frac{16e^{4x}-8e^{3x}+e^{2x}}{(e-1)^4}\right)\right] + \left[\left(\frac{10e^{3x+1}-10e^{2x+1}+e^{x+1}-10e^{3x}+10e^{2x}-e^x}{(e-1)^4}\right)\right]\right]$$

$$= \frac{1}{2}\left[\left(\frac{10e^{3x+1}-10e^{2x+1}+e^{x+1}-18e^{3x}+11e^{2x}-e^x+16e^{4x}}{(e-1)^3}\right)\right]$$

Now by inverse Transform

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k$$

$$S_w(x,t) = S_{w0}(x)t^0 + S_{w1}(x)t^1 + S_{w2}(x)t^2$$

$$S_w(x,t) = \frac{e^x-1}{e-1} + \left(\frac{2e^{2x}-e^x}{(e-1)^2}\right)t^1 + \frac{1}{2}\left[\left(\frac{10e^{3x+1}-10e^{2x+1}+e^{x+1}-18e^{3x}+11e^{2x}-e^x+16e^{4x}}{(e-1)^4}\right)\right]t^2 + \ldots$$

Conclusion:

In the graph X axis represent the values of x and Y-axis represents the saturation of injected liquid $S_w(x)$ of porous media. It is clear from graph that, for each of T, saturation S_w has a increasing tendency along the space co-ordinate axis. Also for each point of X saturation increases as time increases but the rate at which it rises at each point in observed region slows down with increase in time. This shows that the stabilization of the fingers is truly possible with the made for capillary pressure and water saturation.

x	t=0	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5	t=0.6	t=0.7	t=0.8	t=0.9	t=1
0	0	0.16613031	0.332260623	0.498390934	0.664521245	0.83065156	0.996782	1.16291218	1.32904249	1.4951728	1.66130311
0.1	0.061207025	0.26805578	0.474904538	0.681753294	0.888602051	1.09545081	1.3023	1.50914832	1.71599708	1.92284583	2.12969459
0.2	0.128851248	0.38584766	0.642844065	1.41383329	1.156836881	1.41383329	1.67083	1.92782611	2.18482251	2.44181892	2.69881533
0.3	0.203609677	0.52231414	0.841018602	1.159723064	1.478427526	1.79713199	2.115836	2.43454091	2.75324538	3.07194984	3.3906543
0.4	0.286230518	0.68081107	1.075391613	1.469972161	1.864552708	2.25913326	2.653714	3.04829435	3.4428749	3.83745545	4.23203599
0.5	0.377540669	0.86535536	1.353170048	1.840984737	2.328799427	2.81661412	3.304429	3.79224349	4.28005818	4.76787287	5.25568756
0.6	0.478453992	1.08076278	1.683071568	2.285380356	2.887689144	3.48999793	4.092307	4.69461551	5.2969243	5.89923308	6.50154187
0.7	0.589980462	1.33281528	2.075650094	2.81848491	3.561319726	4.30415454	5.046989	5.78982417	6.53265899	7.27549381	8.01832862
0.8	0.713236274	1.62846438	2.543692486	3.458920593	4.374148699	5.28937681	6.204605	7.11983302	8.03506112	8.95028923	9.86551734
0.9	0.849455012	1.97607851	3.102702014	4.229325515	5.355949016	6.48257252	7.609196	8.73581952	9.86244302	10.9890665	12.11569
1	1	2.38574386	3.771487721	5.157231581	6.542975441	7.9287193	9.314463	10.700207	12.0859509	13.4716947	14.8574386

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