Zeros of polar Derivatives of Polynomials

P. Ramulu Department of Mathematics, Govt.Degree College, Wanaparthy, Mahabubnagar,Telangana, India-509103

G. L.Reddy School of Mathematics and Statistics, University of Hyderabad, India-500046.

ABSTRACT *In this paper we extend some existing results on the zeros ofpolar derivatives of polynomials by considering more general coefficient conditions. As special cases the extended results yield much simpler expressions for the upper bounds of zeros than those of the existing results.*

Mathematics Subject Classification: 30C10, 30C15.

KEYWORDS : Zeros of polynomial,Eneström-Kakeya theorem, Polar derivatives.

1.INTRODUCTION

Let $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of a polynomial P(z) of degree n with respect to real number α. The polynomial $D_\alpha P(z)$ is of degree at most n-1 and it generalizes the ordinary derivative in the sense that $\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$. Many results on the location of zeros of polynomials are available in the literature. In literature [3-5] attempts have been made to extend and generalize the Enestrom-Kakeya theorem. Among them the Eneström-Kakeya theorem [1-2] given below is well known in the theory of zero distribution of polynomials.

Theorem A. (Eneström-Kakeyatheorem): LetP (z)=$\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n* such that $0 < a_0 \le a_1 \le a_2 \le \cdots \le a_n$ then all the zeros of P(z) lie in |z|≤1.

The following theorems B and C due to P.Ramulu and G.L. Reddy [6]

Theorem B. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with real coefficients and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α such that $na_0 \le (n-1)a_1 \le (n-2)a_2 \le \cdots \le 3a_{n-3} \le 2a_{n-2} \le a_{n-1}$ *if* $\alpha = 0$ then all the zeros of polar derivative $D_0P(z)$lie in $|z| \le \frac{1}{|a_{n-1}|}[\, a_{n-1} - na_0 + |na_0|\,]$.

Theorem C. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with real coefficients and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α such that
$na_0 \ge (n-1)a_1 \ge (n-2)a_2 \ge \cdots \ge 3a_{n-3} \ge 2a_{n-2} \ge a_{n-1}$ *if* $\alpha = 0$then all the zeros of polar derivative $D_0P(z)$ lie in $|z| \le \frac{1}{|a_{n-1}|}[|na_0| + na_0 - a_{n-1}\,]$.

Here we establish the following results instead of monotone coefficients.

Theorem 1. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with real coefficients and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α such that

$$[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i for\ i = 0,1,\dots,m-1,$$
$$and\ \ (i+1)\alpha a_{i+1} + (n-i)a_i \geq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1}\ \ for\ i = m,\dots,n-2.$$

then all the zeros of polar derivative $D_\alpha = 0P(z)$lie in

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[|\alpha a_1 + na_0| + 2[(m+1)\alpha a_{m+1} + (n-m)a_m] - (n\alpha a_n + a_{n-1} + \alpha a_1 + na_0)].$$

Corollary 1. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with positive real coefficients and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* α such that

$$[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i for\ i = 0,1,\dots,m-1,$$
$$and\ \ (i+1)\alpha a_{i+1} + (n-i)a_i \geq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1}\ \ for\ i = m,\dots,n-2.$$

then all the zeros of polar derivative $D_\alpha P(z)$lie in

$$|z| \leq \frac{1}{n\alpha a_n + a_{n-1}}[2[(m+1)\alpha a_{m+1} + (n-m)a_m] - (n\alpha a_n + a_{n-1})].$$

Corollary 2. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with real coefficients and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number* αsuch that

$$[n-(i+1)]a_{i+1} \geq (n-i)a_i for\ i = 0,1,\dots,m-1,$$
$$and\ \ (n-i)a_i \geq [n-(i+1)]a_{i+1}\ \ for\ i = m,\dots,n-2, if\ \alpha = 0$$

then all the zeros of polar derivative $D_{=0}P(z)$lie in

$$|z| \leq \frac{1}{|a_{n-1}|}[|na_0| + 2(n-m)a_m - (a_{n-1} + na_0)].$$

Remark 1.

By taking $a_i > 0\ fori = 0,1,2,\dots,n-1$,in theorem 1, then it reduces to Corollary 1.

Remark 2. By taking $\alpha = 0$in theorem 1, then it reduces to Corollary 2.

Theorem 2. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with real coefficients and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be a polar derivative of P(z) with respect to real number*αsuch that

$$[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \leq (i+1)\alpha a_{i+1} + (n-i)a_i for\ i = 0,1,\dots,m-1,$$
$$and\ \ (i+1)\alpha a_{i+1} + (n-i)a_i \leq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1}\ \ for\ i = m,\dots,n-2.$$

then all the zeros of polar derivative $D_\alpha P(z)$lie in

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[n\alpha a_n + a_{n-1} - 2[(m+1)\alpha a_{m+1} + (n-m)a_m] + (\alpha a_1 + na_0) + |\alpha a_1 + na_0|].$$

Corollary 3. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with positive real coefficients and* $D_\alpha P(z) = nP(z) + (\alpha - z)P^{'}(z)$ *be a polar derivative of P(z) with respect to real number* α such that

$$[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \le (i+1)\alpha a_{i+1} + (n-i)a_i for\ i = 0,1,\dots,m-1,$$
$$and\ \ (i+1)\alpha a_{i+1} + (n-i)a_i \le [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1}\ \ for\ i = m,\dots,n-2.$$

then all the zeros of polar derivative $D_{=0}P(z)$lie in

$$|z| \le \frac{1}{n\alpha a_n + a_{n-1}}[n\alpha a_n +\ a_{n-1} - 2[(m+1)\alpha a_{m+1} + (n-m)a_m] + 2(\alpha a_1 + na_0)].$$

Corollary 4. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with real coefficients and* $D_\alpha P(z) = nP(z) + (\alpha - z)P^{'}(z)$ *be a polar derivative of P(z) with respect to real number* α such that

$$[n-(i+1)]a_{i+1} \le (n-i)a_i for\ i = 0,1,\dots,m-1,$$
$$and\ \ (n-i)a_i \le [i+2]\alpha a_{i+2}\ for\ i = m,\dots,n-2. if\ \alpha = 0$$

then all the zeros of polar derivative $D_\alpha\ = 0\ P(z)$lie in

$$|z| \le \frac{1}{|n\alpha a_n + a_{n-1}|}[\ a_{n-1} - 2(n-m)a_m + na_0 + |na_0|].$$

Remark 2.

By taking $a_i > 0\ for\ \ i = 0,1,2,\dots,n-1$, in theorem 2, then itreduces to Corollary 3.

Remark 2. By taking $\alpha = 0$in theorem 2, then it reduces to Corollary 4.

2. Proofs of the Theorems

Proof of the Theorem 1.

LetP(z) = $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_{m+1}z^{m+1} + a_m z^m + a_{m-1}z^{m-1} + \cdots + a_2 z^2 + a_1 z + a_0$
be a polynomial of degree n

Let $D_\alpha P(z) = nP(z) + (\alpha - z)P^{'}(z)$ be polar derivative of P(z) with respect to real number α of degree n-1.

$\Rightarrow D_\alpha P(z) = [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2}$
$+[(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + \cdots + [(m+2)\alpha a_{m+2} + (n-m-1)a_{m+1}]z^{m+1} +$
$+[(m+1)\alpha a_{m+1} + (n-m)a_m]z^m + [m\alpha a_m + (n-m+1)a_{m-1}]z^{m-1}$
$+\cdots + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0]$

Let us consider the polynomial Q(z) = (1 − z) $D_\alpha P(z)$ so that

$Q(z) = (1-z)(\ [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2}$

$$+[(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + \cdots + [(m+2)\alpha a_{m+2} + (n-m-1)a_{m+1}]z^{m+1} +$$
$$+[(m+1)\alpha a_{m+1} + (n-m)a_m]z^m + [m\alpha a_m + (n-m+1)a_{m-1}]z^{m-1}$$
$$+ \cdots + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0])$$

$$= \ -[n\alpha a_n + a_{n-1}]z^n + \{\ [n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1}$$
$$+[\ (n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha)a_{n-2} - 3a_{n-3}]\ z^{n-2}$$
$$+ \cdots + [(m+2)\alpha a_{m+2} + \{\ n-(m+1)(1+\alpha)\}a_{m+1} - (n-m)a_m]z^{m+1}$$
$$+\ [(m+1)\alpha a_{m+1} + \{n-m(1+\alpha)\}a_m - (n-m+1)a_{m-1}\ ]z^m +$$
$$[m\alpha a_m + \{n-(m-1)(1+\alpha)\}a_{m-1} - (n-m+2)a_{m-2}]z^{m-1}$$
$$+ \cdots + [3\alpha a_3 + (n-2-2\alpha)a_2 - \ (n-1)a_1]z^2$$
$$+[2\alpha a_2 + (n-1-\alpha)a_1 - na_0]z + [\alpha a_1 + na_0]\ \}$$

Also if $|z| > 1$ then $\frac{\mathbf{1}}{|\boldsymbol{z}|^{\boldsymbol{n-i}}} < 1\ for\ i = 0,1,2,\ldots,n-2.$

Now

$$|Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^n - \{\ |n\alpha a_n\ + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}||z|^{n-1}$$
$$+|(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha)a_{n-2} - 3a_{n-3}||z|^{n-2}$$
$$+ \cdots +|\ (m+2)\alpha a_{m+2} + \{\ n-(m+1)(1+\alpha)\}a_{m+1} - (n-m)a_m\ ||z|^{m+1}$$
$$+|(m+1)\alpha a_{m+1} + \{n-m(1+\alpha)\}a_m - (n-m+1)a_{m-1}||z|^m$$
$$+|m\alpha a_m + \{n-(m-1)(1+\alpha)\}a_{m-1} - (n-m+2)a_{m-2}||z|^{m-1} + \cdots +$$
$$|3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1||z|^2 + |2\alpha a_2 + (n-1-\alpha)a_1 - na_0||z| + |\alpha a_1 + na_0|\ \}$$

$$\geq\ \ |n\alpha a_n + a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|n\alpha\, a_n + a_{n-1}|}\{|\ n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}| +$$
$$\frac{|(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha)a_{n-2} - 3a_{n-3}|}{|z|} + \cdots + \frac{|(m+2)\alpha a_{m+2} + \{\ n-(m+1)(1+\alpha)\}a_{m+1} - (n-m)a_m|}{|z|^{n-m-2}} +$$
$$\frac{|(m+1)\alpha a_{m+1} + \{n-m(1+\alpha)\}a_m - (n-m+1)a_{m-1}|}{|z|^{n-m-1}} + \frac{|m\alpha a_m + \{n-(m-1)(1+\alpha)\}a_{m-1} - (n-m+2)a_{m-2}|}{|z|^{n-m}} + \cdots +$$
$$\frac{|3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1|}{|z|^{n-3}} + \frac{2\alpha a_2 + (n-1-\alpha)a_1 - na_0|}{|z|^{n-2}} + \frac{|\alpha a_1 + na_0|}{|z|^{n-1}}\}\ ]$$

$$\geq |n\alpha a_n + a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\ \{\ |\ n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}|$$
$$+|(n-1)\alpha a_{n-1} + \big(2+2\alpha-n\alpha\big)a_{n-2} - 3a_{n-3}\big| + \cdots +$$
$$|(m+2)\alpha a_{m+2} + \{\ n-(m+1)(1+\alpha)\}a_{m+1} - (n-m)a_m|$$
$$+|(m+1)\alpha a_{m+1} + \{n-m(1+\alpha)\}a_m - (n-m+1)a_{m-1}|$$
$$+|m\alpha a_m + \{n-(m-1)(1+\alpha)\}a_{m-1} - (n-m+2)a_{m-2}| + \cdots +$$
$$\big|3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1\big| + \big|2\alpha a_2 + (n-1-\alpha)a_1 - na_0\big| + |\alpha a_1 + na_0|\ \}]$$

$$\geq |n\alpha a_n + a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{\ \big[2a_{n-2} - (1+\alpha-n\alpha)a_{n-1} - n\alpha a_n\big]$$
$$+[3a_{n-3}\ + \big(2+2\alpha-n\alpha\big)a_{n-2} - (n-1)\alpha a_{n-1}] + \cdots +$$
$$[(n-m)a_m - \{\ n-(m+1)(1+\alpha)\}a_{m+1} - (m+2)\alpha a_{m+2}]$$
$$+[(m+1)\alpha a_{m+1} + \{n-m(1+\alpha)\}a_m - (n-m+1)a_{m-1}]$$
$$+[m\alpha a_m + \{n+(m-1)(1+\alpha)\}a_{m-1} - (n-m+2)a_{m-2}] + \cdots +$$
$$[3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1] + [2\alpha a_2 + (n-1-\alpha)a_1 - na_0] + |\alpha a_1 + na_0|\}]$$

$$\geq |n\alpha a_n + a_{n-1}||z|^{n-1}\Big[\,|z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{|\alpha a_1 + na_0| + 2[(m+1)\alpha a_{m+1} + (n-m)a_m] \\ -(n\alpha a_n + \ a_{n-1} + \alpha a_1 + na_0)\}\Big]$$

$$> 0 \; if \;\; |z| > \frac{1}{|n\alpha a_n + a_{n-1}|}\begin{bmatrix}|\alpha a_1 + na_0| + 2[(m+1)\alpha a_{m+1} + (n-m)a_m] \\ -(n\alpha a_n + \ a_{n-1} + \alpha a_1 + na_0)\end{bmatrix}.$$

This shows that if Q(z)> 0
Provided $|z| > \frac{1}{|n\alpha\, a_n + a_{n-1}|}\begin{bmatrix}|\alpha a_1 + na_0| + 2[(m+1)\alpha a_{m+1} + (n-m)a_m] \\ -(n\alpha a_n + \ a_{n-1} + \alpha a_1 + na_0)\end{bmatrix}.$

Hence all the zeros of Q(z) with |z|> 1 lie in

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}\begin{bmatrix}|\alpha a_1 + na_0| + 2[(m+1)\alpha a_{m+1} + (n-m)a_m] \\ -(n\alpha a_n + \ a_{n-1} + \alpha a_1 + na_0)\end{bmatrix}.$$

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of polar derivative $D_\alpha P(z)$ are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 1.

Proof of the Theorem 2.

LetP(z) $= \; a_n z^n + a_{n-1}z^{n-1} + \cdots + a_{m+1}z^{m+1} + a_m z^m + a_{m-1}z^{m-1} + \cdots + a_2 z^2 + a_1 z + a_0$
be a polynomial of degree n

Let$D_\alpha P(z) = nP(z) + (\alpha - z)P^{'}(z)$ be polar derivative of P(z) with respect to real number α of degree n-1.

$$\Rightarrow D_\alpha P(z) = [n\alpha a_n + a_{n-1}]z^{n-1} + [(n{-}1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} \\ +[(n{-}2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + \cdots + [(m+2)\alpha a_{m+2} + (n-m-1)a_{m+1}]z^{m+1} + \\ +[(m{+}1)\alpha a_{m+1} + (n-m)a_m]z^m + [m\alpha a_m + (n-m+1)a_{m-1}]z^{m-1} \\ + \cdots + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0]$$

Letus consider the polynomial Q(z) $= \; (\,1 - \mathrm{z}\,)\; D_\alpha P(z)$ so that

$$Q(z) = (1-z)([n\alpha a_n + a_{n-1}]z^{n-1} + [(n{-}1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} \\ +[(n{-}2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + \cdots + [(m+2)\alpha a_{m+2} + (n-m-1)a_{m+1}]z^{m+1} + \\ +[(m{+}1)\alpha a_{m+1} + (n-m)a_m]z^m + [m\alpha a_m + (n-m+1)a_{m-1}]z^{m-1} \\ + \cdots + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0])$$

$$= \; -[n\alpha a_n + a_{n-1}]z^n + \{\,[\,n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1} \\ +[\,(n-1)\alpha a_{n-1} + (2{+}2\alpha - n\alpha)a_{n-2} - 3a_{n-3}]\; z^{n-2} \\ + \cdots + [(m+2)\alpha a_{m+2} + \{\,n - (m+1)(1+\alpha)\}a_{m+1} - (n-m)a_m]z^{m+1} \\ + [(m+1)\alpha a_{m+1} + \{n - m(1+\alpha)\}a_m - (n-m+1)a_{m-1}\,]z^m + \\ [m\alpha a_m + \{n-(m-1)(1+\alpha)\}a_{m-1} - (n-m+2)a_{m-2}]z^{m-1}$$

$$+\cdots+[3\alpha a_3+(n-2-2\alpha)a_2-\ (n-1)a_1]z^2$$
$$+[2\alpha a_2+(n-1-\alpha)a_1-na_0]z+[\alpha a_1+na_0]\,\}$$

Also if $|z|>1$ then $\frac{\mathbf{1}}{|\boldsymbol{z}|^{\boldsymbol{n-i}}}<1\ for\ i=0,1,2,\dots,n-2.$

Now

$$|Q(z)|\geq|n\alpha a_n+a_{n-1}||z|^n-\{\,|n\alpha a_n\ +(1+\alpha-n\alpha)a_{n-1}-2a_{n-2}||z|^{n-1}$$
$$+|(n-1)\alpha a_{n-1}+(2{+}2\alpha-n\alpha)a_{n-2}-3a_{n-3}||z|^{n-2}$$
$$+\cdots+|\,(m+2)\alpha a_{m+2}+\{\,n-(m+1)(1+\alpha)\}a_{m+1}-(n-m)a_m\,||z|^{m+1}$$
$$+|(m+1)\alpha a_{m+1}+\{n-m(1+\alpha)\}a_m-(n-m+1)a_{m-1}||z|^m$$
$$+|m\alpha a_m+\{n-(m-1)(1+\alpha)\}a_{m-1}-(n-m+2)a_{m-2}||z|^{m-1}+\cdots+$$
$$|3\alpha a_3+(n-2-2\alpha)a_2-(n-1)a_1||z|^2+|2\alpha a_2+(n-1-\alpha)a_1-na_0||z|+|\alpha a_1+na_0|\}$$

$$\geq\ \ |n\alpha a_n+a_{n-1}||z|^{n-1}[\,|z|-\frac{1}{|n\alpha\,a_n+a_{n-1}|}\{|\,n\alpha a_n+(1+\alpha-n\alpha)a_{n-1}-2a_{n-2}|+$$
$$\frac{|(n-1)\alpha a_{n-1}+(2+2\alpha-n\alpha)a_{n-2}-3a_{n-3}|}{|z|}+\cdots+\frac{|(m+2)\alpha a_{m+2}+\{\,n-(m+1)(1+\alpha)\}a_{m+1}-(n-m)a_m|}{|z|^{n-m-2}}+$$
$$\frac{|(m+1)\alpha a_{m+1}+\{n-m(1+\alpha)\}a_m-(n-m+1)a_{m-1}|}{|z|^{n-m-1}}+\frac{|m\alpha a_m+\{n-(m-1)(1+\alpha)\}a_{m-1}-(n-m+2)a_{m-2}|}{|z|^{n-m}}+\cdots+$$
$$\frac{|3\alpha a_3+(n-2-2\alpha)a_2-(n-1)a_1|}{|z|^{n-3}}+\frac{2\alpha a_2+(n-1-\alpha)a_1-na_0|}{|z|^{n-2}}+\frac{|\alpha a_1+na_0|}{|z|^{n-1}}\}\,]$$

$$\geq|n\alpha a_n+a_{n-1}||z|^{n-1}[\,|z|-\frac{1}{|n\alpha a_n+a_{n-1}|}\ \{\,|\,n\alpha a_n+(1+\alpha-n\alpha)a_{n-1}-2a_{n-2}|$$
$$+|(n-1)\alpha a_{n-1}+\big(2{+}2\alpha-n\alpha\big)a_{n-2}-3a_{n-3}\big|+\cdots+$$
$$|(m+2)\alpha a_{m+2}+\{\,n-(m+1)(1+\alpha)\}a_{m+1}-(n-m)a_m|$$
$$+|(m+1)\alpha a_{m+1}+\{n-m(1+\alpha)\}a_m\ -(n-m+1)a_{m-1}|$$
$$+|m\alpha a_m+\{n-(m-1)(1+\alpha)\}a_{m-1}-(n-m+2)a_{m-2}|+\cdots+$$
$$\big|3\alpha a_3+(n-2-2\alpha)a_2-(n-1)a_1\big|+\big|2\alpha a_2+(n-1-\alpha)a_1-na_0\big|+|\alpha a_1+na_0|\,\}]$$

$$\geq|n\alpha a_n+a_{n-1}||z|^{n-1}[\,|z|-\frac{1}{|n\alpha a_n+a_{n-1}|}\{\,\big[n\alpha a_n+(1+\alpha-n\alpha)a_{n-1}-2a_{n-2}\big]$$
$$+[\,(n-1)\alpha a_{n-1}+\big(2{+}2\alpha-n\alpha\big)a_{n-2}-3a_{n-3}\,]+\cdots+$$
$$[(m+2)\alpha a_{m+2}+\{\,n-(m+1)(1+\alpha)\}a_{m+1}-(n-m)a_m]$$
$$+[(n-m+1)a_{m-1}-\{n-m(1+\alpha)\}a_m-(m+1)\alpha a_{m+1}]$$
$$+[(n-m+2)a_{m-2}-\{n-(m-1)(1+\alpha)\}a_{m-1}-m\alpha a_m]+\cdots+$$
$$[(n-1)a_1-(n-2-2\alpha)a_2-3\alpha a_3]+[na_0-(n-1-\alpha)a_1-2\alpha a_2]+|\alpha a_1+na_0|\}]$$

$$\geq|n\alpha a_n+a_{n-1}||z|^{n-1}[\,|z|-\frac{1}{|n\alpha a_n+a_{n-1}|}\{n\alpha a_n+\ a_{n-1}-2[(m{+}1)\alpha a_{m+1}+(n-m)a_m]$$
$$+(\alpha a_1+na_0)+|\alpha a_1+na_0|\,\}]$$

$$>0\,if\ \ |z|>\frac{1}{|n\alpha a_n+a_{n-1}|}\begin{bmatrix}n\alpha a_n+\ a_{n-1}-2[(m{+}1)\alpha a_{m+1}+(n-m)a_m]\\ +(\alpha a_1+na_0)+|\alpha a_1+na_0|\end{bmatrix}.$$

This shows that if Q(z)> 0

provided$|z| > \frac{1}{|n\alpha\, a_n + a_{n-1}|}\begin{bmatrix} n\alpha a_n + \ a_{n-1} - 2[(m+1)\alpha a_{m+1} + (n-m)a_m] \\ +(\alpha a_1 + na_0) + |\alpha a_1 + na_0| \end{bmatrix}.$

Hence all the zeros of Q(z) with |z|> 1 lie in

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}\begin{bmatrix} n\alpha a_n + \ a_{n-1} - 2[(m+1)\alpha a_{m+1} + (n-m)a_m] \\ +(\alpha a_1 + na_0) + |\alpha a_1 + na_0| \end{bmatrix}.$$

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of polar derivative$D_\alpha\, P(z)$ are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 2.

REFERENCES

[1]. G.Eneström,Remarqueesur un théorèmerelatif aux racines de l'equationa_n+...+a_0=0 oü tous les coefficient sont et positifs, Tôhoku Math.J 18 (1920),34-36. | [2]. S.KAKEYA, On the limits of the roots of an alegebraic equation with positive coefficient, TôhokuMath.J 2 (1912-1913),140-142. | [3] P.Ramulu, G.L .Reddy,On the Enestrom-Kakeya theorem. International Journal of Pure and Applied Mathematics, Vol. 102 No.4, 2015. (Up Coming issue). | [4]Gulshan Singh, Wali Mohammad Shah, Yash Paul,Inequalities for the Polar Derivative of a Polynomial,Advances in Pure Mathematics, 2011, 1, 23-27. | [5] P.Ramulu, Some Generalization of Eneström-Kakeya Theorem,International Journal of Mathematics and Statistics Invention (IJMSI),Volume 3 Issue 2 || February. 2015 || PP-52-59. | [6] P. Ramulu and G.L. Reddy ,On The Zeros of Polar Derivatives,International Journal of RecentResearch in Mathematics Computer Science and Information Technology(IJRRMCSIT) Vol.2, Issue 1,pp: (143-145), Month: April 2015 – September 2015.