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Two Parameter Rayleigh Distribution and Asymmetric Loss Functions: a Bayesian Perspective

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ABSTRACT *We shall note that the phenomenon of change point is observed in several situations when we study a life time model. At some point of time, we observe instability in the sequence of life time under observation and study. Our study is mainly focused to find that change point, where we need to find out at what time and at which point, the changes begin to occur. This entire phenomenon is called the change point inference problem. Here, we have clearly proposed the Bayesian Estimation Method as a strong and valid alternative to the method of Classical Estimation. Our purpose is to study the Two Parameter Rayleigh Model with a change point from the Bayesian point of view using Asymmetric Loss Functions.*

KEYWORDS : Two Parameter Rayleigh Distribution, Two Parameter Rayleigh Life Time Model, Scale Parameter, Location Parameter, Bayes Estimates, Change Point, Asymmetric Loss Functions, Linex Loss Function, General Entropy Loss Function

1. INTRODUCTION

It was Lord Rayleigh who introduced the Rayleigh Distribution in the year 1880. It was studied and considered for the first time when there was some problem related to the acoustics field. The hazard function is an increasing function of time in case of Rayleigh Distribution which is a very significant characteristic to be considered as far as our study is concerned. According to this characteristic, the aging process or the deterioration process of the equipments or items start occurring in a very intense manner. It follows the Rayleigh Law when the failure times are distributed and above mentioned process takes place. We must not forget to mention the invaluable contributions of several researchers who studied the concept of Rayleigh Distribution and gave the conclusions on the basis of their studies. Among those research scholars, Johnson, Kotz and Balakrishnan studied the Rayleigh Law and made a study related to the excellent exposure of the Rayleigh Distribution in the year 1994. Their research work was studied and further carried out by the team of three research scholars named Abd-Elfattah, Hassan and Ziedean in the year 2006. The research was taken further by the joint efforts of Dey and Das in the year 2007 and Dey gave some latest results and conclusions in the year 2009 on the basis of the research work carried out earlier by the above mentioned scholars in the field of Statistics, which includes estimations, predictions and inferential issues for One Parameter Rayleigh Distribution. The Compound Rayleigh Model with a unimodal hazard function was obtained by Mostert, Roux and Bekker in the year 1991 and the Generalized Compound Rayleigh Model was studied by them in the year 2001 from the Bayesian perspective. The main application of the Compound Rayleigh Distribution is for the modelling of the survival times of patients which show the characteristics of a random hazard rate. Here, we shall consider the case of Generalization of the Two Parameter Rayleigh Life Time Distribution. The survival distribution and hazard function of this distribution are given as under:

$$f(x_i, \lambda, \mu) = 2\lambda (x_i - \mu)\, e^{-\lambda (x_i - \mu)^2}; \; x_i > \mu, \lambda > 0 \text{ and } i = 1, 2, 3, \ldots, m$$

$$S(t) = e^{-\lambda (t-\mu)^2}, \qquad t > \mu$$

$$h(t) = 2\lambda (t - \mu), \qquad t > 0 \qquad (1)$$

As stated earlier, we shall clearly note that the phenomenon of change point is observed in several situations when we study a life time model. At some point of time, we observe instability in the sequence of life time under observation and study. Our study is mainly focused to find that change point, where we need to find out at what time and at which point, the changes begin to occur. This entire phenomenon is called the change point inference problem. Here, we have clearly proposed the Bayesian Estimation Method as a strong and valid alternative to the method of Classical Estimation. Thus, the purpose is to study the Two Parameter Rayleigh Model with a change point from the Bayesian point of view.

Here, we have proposed a change point model related to the Two Parameter Rayleigh Distribution, where λ is a scale parameter and μ is a location parameter. In the next section, we have obtained the Bayes Estimates and posterior densities of λ_1, λ_2 and 'm'. Then we have derived the Bayes Estimates of $\lambda_1, \lambda_2, S_1(t), S_2(t), h_1(t), h_2(t)$ and 'm' under asymmetric loss functions. After that, we have generated a numerical example and then we have given conclusions on the basis of the numerical study.

2. PROPOSED CHANGE POINT MODEL

Let $X_1, X_2, X_3, \ldots, X_n$ (n ≥ 3) be a sequence of random lifetimes. Let first 'm' observations be coming from Two Parameter Rayleigh Distribution with the parameters $(\lambda_{1,}\mu)$. So the probability density function is given by:

$$f\,(x_i\,, \lambda_1, \mu) = 2\lambda_1\,(x_i - \mu)\,e^{-\lambda_1\,(x_i - \mu)^2}; \qquad (2)$$

where $x_i > \mu, \lambda_1 > 0$ and i= 1, 2, 3,…,m with the survival distribution function $S_1(t)$ and hazard function $h_1(t)$given by,

$$S_1(t) = e^{-\lambda_1\,(t-\mu)^2} \quad ; \qquad t > \mu \qquad (3)$$

$$h_1(t) = 2\lambda_1\,(t - \mu)\ ; \qquad t > 0 \qquad (4)$$

Later n-m observations are coming from the Two Parameter Rayleigh Distribution $(\lambda_{2,}\mu)$ probability density function is given by,

$$f(x_i\,, \lambda_2, \mu) = 2\lambda_2\,(x_i - \mu)\,e^{-\lambda_2(x_i - \mu)^2}; \qquad x_i > \mu, \lambda_2 > 0 \text{ and } i=m+1,\ldots, n \qquad (5)$$

with the survival distribution function $S_2(t)$ and hazard function $h_2(t)$given by,

$$S_2(t) = e^{-\lambda_2\,(t-\mu)^2} \quad ; \qquad t > 0 \qquad (6)$$

$$h_2(t) = 2\lambda_2\,(t-\mu) \quad ; \qquad t > 0 \qquad (7)$$

For the given sample information T= $(T_1,\ldots, T_m, T_{m+1}\,.\,.\,.\,, T_n)$, the likelihood function will be as under:

$$L(\lambda_1,\lambda_2,\mu,m|\underline{T}) = 2^n\,U\,{\lambda_1}^m e^{-\lambda_1 T_1}\;{\lambda_2}^{n-m} e^{-\lambda_2 T_2} \qquad (8)$$

where,

$T_1 = T_1(m) = \sum_{i=1}^{m}(x_i - \mu)^2$,

$T_2 = T_2(m) = \sum_{i=m+1}^{n}(x_i - \mu)^2$,

$$U = \prod_{i=1}^{n}(x_i - \mu) \qquad (9)$$

3. POSTERIOR DENSITIES

We suppose the marginal prior distribution of 'm' to be discrete uniform over the set

{1, 2, 3,..., n-1}.

$$g(m) = \frac{1}{n-1} \qquad (10)$$

We also suppose a discrete prior distribution on the parameter μ considering the reference of the research work of Soland carried out in the year 1969. Further, we make an assumption that let the parameter μ be restricted to finite number of values say, $\mu_1, \mu_2, \ldots, \mu_w$ in the interval (0,1).

Pr $(\mu)=\xi_j$,

$$\sum_{j=1}^{w}\xi_j = 1, \quad 0 \le \xi_j \le 1, \quad j=1, 2,\ldots, w \qquad (11)$$

Let us now suppose the conditional gamma prior on λ_1 and λ_2 given $\mu=\xi_j$ where,

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$j = 1, 2, \ldots, n$ which leads to tractable mathematics, viz.

$$g(\lambda_1 | \xi_j) = \frac{{b_1}^{a_1}}{\Gamma a_1} {\lambda_1}^{a_1 - 1} e^{-b_1 \lambda_1}, \qquad \lambda_1 > 0,\ a_1, b_1 > 0$$

$$g(\lambda_2 | \xi_j) = \frac{{b_2}^{a_2}}{\Gamma a_2} {\lambda_2}^{a_2 - 1} e^{-b_2 \lambda_2} \qquad \lambda_2 > 0,\ a_2, b_2 > 0 \tag{12}$$

Let the prior information be given respectively in terms of prior means $\mu_1 and\ \mu_2$ and coefficient of variations $\Phi_1 and\ \Phi_2$. Then, we have:

$$\mu_1 = \frac{a_1}{b_1},$$

$$\Phi_1 = \frac{1}{\sqrt{a_1}},$$

$$\mu_2 = \frac{a_2}{b_2} \text{ and}$$

$$\Phi_2 = \frac{1}{\sqrt{a_2}} \tag{13}$$

Thus, if we have prior knowledge of μ_1, μ_2, Φ_1 and Φ_2, then gamma parameter can be

obtained by $a_1 = \frac{1}{{\Phi_1}^2}$,

$$b_1 = \frac{1}{\mu_1 {\Phi_1}^2},$$

$$a_2 = \frac{1}{{\Phi_1}^2} \text{ and } b_2 = \frac{1}{\mu_2 {\Phi_2}^2} \tag{14}$$

Hence, the joint prior density will be:

$$g(\lambda_1, \lambda_2, \mu, m) = \frac{1}{n-1} \xi_j \frac{{b_1}^{a_1}}{\Gamma a_1} \frac{{b_2}^{a_2}}{\Gamma a_2} {\lambda_1}^{a_1 - 1} e^{-b_1 \lambda_1} {\lambda_2}^{a_2 - 1} e^{-b_2 \lambda_2} \tag{15}$$

The joint posterior density of parameters $\lambda_1, \lambda_2, \mu$ and $'m'$ is obtained using the likelihood function and the joint prior density of the parameters as under:

$$g(\lambda_1,\lambda_2,\mu,m|T_{1j},T_{2j},\xi_j) = \frac{L(\lambda_1,\lambda_2,\mu,m|T_{1j},T_{2j},\xi_j)g(\lambda_1,\lambda_2,\mu,m)}{h(T)}$$

$$=k_1\xi_j c^n U_j \lambda_1^{a_1+m-1}e^{-\lambda_1(T_1+b_1)}\lambda_2^{a_2+n-m-1}e^{-\lambda_2(T_2+b_2)}/h(\underline{T}) \quad (16)$$

where, $k_1=\frac{1}{n-1}\frac{{b_1}^{a_1}}{\Gamma a_1}\frac{{b_2}^{a_2}}{\Gamma a_2}$, $U_j = \prod_{i=1}^{n}(x_i - \mu_j)$, $T_{1j} = \sum_{i=1}^{m}(x_i - \mu_j)^2$

and $T_{2j} = \sum_{i=m+1}^{n}(x_i - \mu_j)^2$ (17)

and $h(\underline{T})$ is the marginal posterior density of $\underline{T}$.

$$h(\underline{T}) = \sum_{m=1}^{n-1}\sum_{j=1}^{w}\int_0^\infty\int_0^\infty k_1\xi_j\, U_j\, \lambda_1^{a_1+m-1}e^{-\lambda_1(T_{1j}+b_1)}\lambda_2^{a_2+n-m-1}e^{-\lambda_2(T_{2j}+b_2)}$$

$$=\sum_{m=1}^{n-1}\sum_{j=1}^{w}k_1\xi_j\, U_j\int_0^\infty \lambda_1^{a_1+m-1}e^{-\lambda_1(T_{1j}+b_1)}d\lambda_1\int_0^\infty \lambda_2^{a_2+n-m-1}e^{-\lambda_2(T_{2j}+b_2)}\,d\lambda_2$$

$$= \sum_{m=1}^{n-1}\sum_{j=1}^{w}k_1\xi_j\, U_j\frac{\Gamma m+a_1}{(T_{1j}+b_1)^{m+a_1}}\frac{\Gamma n-m+a_2}{(T_{2j}+b_2)^{n-m+a_2}} \quad (18)$$

We shall apply the discrete version of Bayes theorem to obtain the marginal posterior probability distribution of the $\mu = \mu_j$ as under:

$$P_j = P_r\left(\mu = \mu_j|T_j\right) \propto$$

$$\sum_{m=1}^{n-1}k_1\xi_j U_j\int_0^\infty \lambda_1^{a_1+m-1}e^{-\lambda_1(T_{1j}+b_1)}d\lambda_1\int_0^\infty \lambda_2^{a_2+n-m-1}e^{-\lambda_2(T_{2j}+b_2)}\,d\lambda_2 h^{-1}(\underline{T})$$

$$g\left(\lambda_1, \lambda_2 \,|\mu_j, \underline{\mathrm{T}}\right) = \sum_{m=1}^{n-1} \mathrm{k}_1 \xi_j\, U_j\, \lambda_1{}^{a_1+\mathrm{m}-1} e^{-\lambda_1(T_{1j}+b_1)} \lambda_2{}^{\mathrm{a}_2+\mathrm{n}-\mathrm{m}-1} e^{-\lambda_2(T_{2j}+b_2)} h^{-1}(\underline{\mathrm{T}})$$

Marginal posterior density $g\left(\lambda_1 \middle| \mu_j, \underline{\mathrm{T}}\right)$ and $g(\lambda_2|\mu_j, \underline{\mathrm{T}})$ will be:

$$g(\lambda_1|\mu_j, \underline{\mathrm{T}}) = \sum_{m=1}^{n-1} \int_0^{\infty} g(\lambda_1, \lambda_2|\mu_j, \underline{\mathrm{T}})\, \mathrm{d}\lambda_2$$

$$= \sum_{m=1}^{n-1} \mathrm{k}_1 \xi_j\, U_j \lambda_1{}^{a_1+\mathrm{m}-1} e^{-\lambda_1(T_{1j}+b_1)} \frac{\Gamma \mathrm{n}-\mathrm{m}+\mathrm{a}_2}{(T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2}} h^{-1}(\underline{\mathrm{T}}) \qquad (20)$$

and $g(\lambda_2|\mu_j, \underline{\mathrm{T}}) = \sum_{m=1}^{n-1} \int_0^{\infty} g_1(\lambda_1, \lambda_2|\mu_j, \underline{\mathrm{T}})\, \mathrm{d}\lambda_1$

$$= \sum_{m=1}^{n-1} \mathrm{k}_1 \xi_j\, U_j \lambda_2{}^{\mathrm{a}_2+\mathrm{n}-\mathrm{m}-1} e^{-\lambda_2(T_{2j}+b_2)} \frac{\Gamma \mathrm{m}+a_1}{(T_{1j}+b_1)^{\mathrm{m}+a_1}} h^{-1}(\underline{\mathrm{T}}) \qquad (21)$$

Combining (19) and (20), marginal posterior density of λ_1 say $g(\lambda_1|\underline{\mathrm{T}})$ will be as under:

$$g(\lambda_1|\underline{\mathrm{T}}) = \sum_{m=1}^{n-1} \sum_{\mathrm{j}=1}^{\mathrm{w}} \frac{\boldsymbol{P}_j (T_{1j}+b_1)^{\mathrm{m}+a_1}\ \lambda_1{}^{a_1+\mathrm{m}-1} e^{-\lambda_1(T_{1j}+b_1)}}{\Gamma \mathrm{m}+a_1} \qquad (22)$$

Combining (19) and (21), marginal posterior density of λ_1 say $g(\lambda_2|\underline{\mathrm{T}})$ will be:

$$g(\lambda_2|\underline{\mathrm{T}}) = \sum_{m=1}^{n-1} \sum_{\mathrm{j}=1}^{\mathrm{w}} \frac{\boldsymbol{P}_j (T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\ \lambda_2{}^{\mathrm{a}_2+\mathrm{n}-\mathrm{m}-1} e^{-\lambda_2(T_{2j}+b_2)}}{\Gamma \mathrm{n}-\mathrm{m}+\mathrm{a}_2} \qquad (23)$$

The marginal posterior density of change point 'm' will be:

$$g\left(\mathrm{m} \middle| \mu_j, \underline{\mathrm{T}}\right) = B(m)/h(\underline{\mathrm{T}}) \text{ where j=1, 2,…,}w \qquad (24)$$

where $B(m) = \mathrm{k}_1 \xi_j U_j \int_0^{\infty} \lambda_1{}^{a_1+\mathrm{m}-1} e^{-\lambda_1(T_{1j}+b_1)}\, d\lambda_1 \int_0^{\infty} \lambda_2{}^{\mathrm{a}_2+\mathrm{n}-\mathrm{m}-1} e^{-\lambda_2(T_{2j}+b_2)} d\lambda_2$

$$= \mathrm{k}_1 \xi_j U_j \quad \frac{\Gamma \mathrm{m}+a_1 \Gamma \mathrm{n}-\mathrm{m}+\mathrm{a}_2}{(T_{1j}+b_1)^{\mathrm{m}+a_1} (T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2}} \qquad (25)$$

We get the marginal posterior density of the change point 'm' on combining (19) and (24) as

$$g(\mathrm{m}\,|\underline{\mathrm{T}}) = \sum_{j=1}^{W} P_j B\,(m)/h(\underline{\mathrm{T}}) \tag{26}$$

4. BAYES ESTIMATES UNDER ASYMMETRIC LOSS FUNCTIONS

In this section, we have obtained the Bayes Estimates of the change point, survival times, hazard rates and parameters λ_1, λ_2. We shall use the asymmetric loss function known as the Linex Loss Function for getting those estimates as under.

Minimizing expected loss function $E_m\,[L_4\,(\mathrm{m, d})]$ and using posterior distribution, we get the Bayes Estimates of 'm' under Linex Loss Function by means of the nearest integer value, say m_L^*, as under:

$$m_L^* = -1/q_1\,.\,ln[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j e^{-q_1 m} B(m)\,/h(\underline{\mathrm{T}})] \tag{27}$$

where $B(m)$ and $h(\underline{\mathrm{T}})$ are same as obtained in (18) and (25) respectively.

Now, here also minimizing expected loss function $E_{\lambda_1}[L_4(\lambda_1, \mathrm{d})]$ and using the posterior distribution, we can get the Bayes Estimates of λ_1 under Linex Loss Function as shown below:

$$\lambda_{1L}^* = -\frac{1}{q_1}\ln[E(e^{-\lambda_1 q_1})]$$

$$= -\frac{1}{q_1}\ln[\int_0^\infty g(\lambda_1|T_{1j}, T_{2j})\, P_j e^{-\lambda_1 q_1}\, \mathrm{d}\lambda_1] =$$

$$-\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{1j}+b_1)^{\mathrm{m+a_1}}}{\Gamma\mathrm{m+a_1}}\int_0^\infty \lambda_1{}^{\mathrm{a_1+m-1}} e^{-\lambda_1(T_{1j}+b_1+q_1)}\,\mathrm{d}\lambda_1] =$$

$$-\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{1j}+b_1)^{\mathrm{m+a_1}}\Gamma\mathrm{m+a_1}}{\Gamma\mathrm{m+a_1}(T_{1j}+b_1+q_1)^{\mathrm{m+a_1}}}]$$

$$= -\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{1j}+b_1)^{\mathrm{m+a_1}}}{(T_{1j}+b_1+q_1)^{\mathrm{m+a_1}}} \tag{28}$$

Now, minimizing expected loss function $E_{\lambda_2}[L_4(\lambda_2, \mathrm{d})]$ and using posterior distribution, we can get the Bayes Estimates of λ_2 under Linex Loss function as under:

$$\lambda_{2L}^* = -\frac{1}{q_1}\ln[E(e^{-\lambda_2 q_1})]$$

$$= \frac{1}{q_1}\ln[\int_0^\infty g(\lambda_2|T_{1j}, T_{2j})\, P_j e^{-\lambda_2 q_1}\, \mathrm{d}\lambda b_2] \qquad =$$

$$-\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{2j}+b_2)^{\mathrm{n-m+a}}}{\Gamma\mathrm{n-m+a_2}}\int_0^\infty {\lambda_2}^{\mathrm{a_2+n-m-1}} e^{-\lambda_2(T_{2j}+b_2+q_1)}\, \mathrm{d}\lambda_2]$$

$$= -\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{2j}+b_2)^{\mathrm{n-m+a}}}{\Gamma\mathrm{n-m+a_2}} \frac{\Gamma\mathrm{n-m+a_2}}{(T_{2j}+b_2+q_1)^{\mathrm{n-m+a}}}]$$

$$= -\frac{1}{q_1}\ln\left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{2j}+b_2)^{\mathrm{n-m+a}}}{(T_{2j}+b_2+q_1)^{\mathrm{n-m+a}}}\right] \qquad (29)$$

We can get the Bayes Estimates $\mathrm{s}_{1\mathrm{L}}^*(\mathrm{t}), \mathrm{s}_{2\mathrm{L}}^*(\mathrm{t}), \mathrm{h}_{1\mathrm{L}}^*(\mathrm{t}), \mathrm{h}_{2\mathrm{L}}^*(\mathrm{t})$ of $s_1(\mathrm{t}), s_2(\mathrm{t}), h_1(\mathrm{t})$ and $h_2(\mathrm{t})$ under Linex Loss Function respectively as under:

$$\mathrm{s}_{1\mathrm{L}}^* = -\frac{1}{q_1}\ln[\int_0^\infty \sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j e^{-q_1}\, g(\lambda_1|T_{1j}, T_{2j})\, \mathrm{d}\lambda_1]$$

This can be rewritten as using the exponential series in following manner:

$$\mathrm{s}_{1\mathrm{L}}^* = -\frac{1}{q_1}\ln\left[\int_0^\infty \sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty} P_j \frac{(-q_1)^k}{k!} g(\lambda_1|T_j)\, \mathrm{d}\lambda_1\right]$$

$$= -\frac{1}{q_1}\ln\left[\int_0^\infty \sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{\mathrm{k}=0}^{\infty} \mathrm{P_j} \frac{(-q_1)^k}{k!} \frac{(T_{1j}+b_1)^{\mathrm{m+a_1}}\, {\lambda_1}^{\mathrm{a_1+m-1}} e^{-\lambda_1(T_1 j+b_1)}}{\Gamma\mathrm{m+a_1}}\right]$$

$$= -\frac{1}{q_1}\ln\left[\int_0^\infty \sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty} P_j \frac{(-q_1)^k}{k!} \frac{(T_{1j}+b_1)^{\mathrm{m+a_1}}\, {\lambda_1}^{\mathrm{a_1+m-1}}}{\Gamma\mathrm{m+a_1}}\right]$$

$$= -\frac{1}{q_1}\ln\left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty} P_j \frac{(-q_1)^k (T_{1j}+b_1)^{\mathrm{m+a_1}}\Gamma\mathrm{m+a_1}}{k!\left((T_{1j}+b_1)\right)^{\mathrm{m+a_1}}\Gamma\mathrm{m+a_1}}\right]$$

$$= -\frac{1}{q_1}\ln\left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty}P_j\frac{(-q_1)^k}{k!}\left(1+\frac{1}{(T_{1j}+b_1)}\right)^{-(\mathrm{m}+\mathrm{a}_1)}\right] \tag{30}$$

$$\mathrm{s}_{2\mathrm{L}}^{*} = -\frac{1}{q_1}\ln\left[\int_0^{\infty}\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty}P_j\, e^{-q_1}\, g(\lambda_2|T_{1j},T_{2j})\,\mathrm{d}\lambda_2\right]$$

This can be rewritten using the exponential series as under:

$$\mathrm{s}_{2\mathrm{L}}^{*} = -\frac{1}{q_1}\ln\left[\int_0^{\infty}\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty}P_j\frac{(-q_1)^k}{k!}\, g(\lambda_2|T_j)\,\mathrm{d}\lambda_2\right]$$

$$= -\frac{1}{q_1}\ln\left[\int_0^{\infty}\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty}P_j\frac{(-q_1)^k}{k!}\frac{(T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\,{\lambda_2}^{\mathrm{a}_2+\mathrm{n}-\mathrm{m}-1}e^{-\lambda_2(T_{2j}+b_2)}}{\Gamma\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\,\mathrm{d}\lambda_2\right]$$

$$= -\frac{1}{q_1}\ln\left[\int_0^{\infty}\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty}P_j\frac{(-q_1)^k(T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\,{\lambda_2}^{\mathrm{a}_2+\mathrm{n}-\mathrm{m}-1}}{k!\,\Gamma\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\,\mathrm{d}\lambda_2\right]$$

$$= -\frac{1}{q_1}\ln\left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty}P_j\frac{(-q_1)^k(T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\Gamma\mathrm{n}-\mathrm{m}+\mathrm{a}_2}{k!(T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\Gamma\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\right]$$

$$= -\frac{1}{q_1}\ln\left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}\sum_{k=0}^{\infty}P_j\frac{(-q_1)^k}{k!}\left(1+\frac{1}{(T_{2j}+b_2)}\right)^{-(\mathrm{n}-\mathrm{m}+\mathrm{a}_2)}\right] \tag{31}$$

Now, $\mathrm{h}_{1\mathrm{L}}^{*} = -\frac{1}{q_1}\ln[\int_0^{\infty}\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}P_j\ e^{-q_1}\, g(\lambda_1|T_{1j},T_{2j})\,\mathrm{d}\lambda_1$

$$= -\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}P_j\, e^{-q_1}\frac{(T_{1j}+b_1)^{\mathrm{m}+\mathrm{a}_1}}{\Gamma\mathrm{m}+\mathrm{a}_1}\int_0^{\infty}{\lambda_1}^{\mathrm{a}_1+\mathrm{m}-1}e^{-\lambda_1(T_{1j}+b_1)}\,\mathrm{d}\lambda_1]$$

$$= -\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}P_j\frac{(T_{1j}+b_1)^{\mathrm{m}+\mathrm{a}_1}}{\Gamma\mathrm{m}+\mathrm{a}_1}\int_0^{\infty}{\lambda_1}^{\mathrm{a}_1+\mathrm{m}-1}e^{-\lambda_1[T_{1j}+b_1+q_1]}\,\mathrm{d}\lambda_1]$$

$$= -\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}}P_j\frac{(T_{1j}+b_1)^{\mathrm{m}+\mathrm{a}_1}\Gamma\mathrm{m}+\mathrm{a}_1}{\Gamma\mathrm{m}+\mathrm{a}_1[T_{1j}+b_1]^{\mathrm{m}+\mathrm{a}_1}}]$$

$$=-\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{j=1}^{w}P_j\left[\frac{q_1}{(T_{1j}+b_1)}+1\right]^{-(m+a_1)}] \quad (32)$$

Similarly, we have

$$h_{2L}^{*}=-\frac{1}{q_1}\ln[\int_0^{\infty}\sum_{m=1}^{n-1}\sum_{j=1}^{w}P_j\ e^{-q_1}\,g(\lambda_2|T_{1j},T_{2j})\,d\lambda_2$$

$$=-\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{j=1}^{w}P_j\,e^{-q_1}\,\frac{(T_{2j}+b_2)^{n-m+a_2}}{\Gamma n-m+a_2}\int_0^{\infty}\lambda_2{}^{n-m+a_2}e^{-\lambda_2(T_{2j}+b_2)}\ d\lambda_2]$$

$$=-\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{j=1}^{w}P_j\,\frac{(T_{2j}+b_2)^{n-m+a_2}}{\Gamma n-m+a_2}\int_0^{\infty}\lambda_2{}^{n-m+a_2}e^{-\lambda_2[(T_{2j}+b_2)+q_1]}\,d\lambda_2]$$

$$=-\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{j=1}^{w}P_j\,\frac{(T_{2j}+b_2)^{n-m+a_2}\Gamma n-m+a_2}{\Gamma n-m+a_2[T_{2j}+b_2)]^{n-m+a_2}}$$

$$=-\frac{1}{q_1}\ln[\sum_{m=1}^{n-1}\sum_{j=1}^{w}P_j\left[\frac{q_1}{(T_{2j}+b_2)}+1\right]^{-(n-m+a_2)}] \quad (33)$$

Similarly, we can obtain the Bayes Estimates m_E^* of 'm' using General Entropy Loss Function as under.

Here also, minimizing the expectation $E_m[L_5$ (m, d)] and using posterior distributions, we can get the Bayes Estimate by the means of the nearest integer value, say m_E^* , as under using General Entropy Loss Function.

$$m_E^* = \left[E[m^{-q_3}]\right]^{-1/q_3}$$

$$=\left[\sum_{m=1}^{n-1}\sum_{j=1}^{w}P_j\,m^{-q_3}B\,(m)/\,h(\underline{T})\,\right]^{-1/q_3} \quad (34)$$

where $B(m)$ and $h(\underline{T})$ are same as obtained in (18) and (25) respectively.

Now, minimizing the expectation $E_{\lambda_1}[L_5(\lambda_1, \mathrm{d})]$ and using posterior distributions (22), we get Bayes Estimates of λ_1 using General Entropy Loss Function as under:

$$\lambda_{1E}^{*} = [E(\lambda_1{}^{-q_3})]^{-\frac{1}{q_3}}$$

$$=[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \int_0^{\infty} g\ (\lambda_1|T_{1j}, T_{2j})\lambda_1{}^{-q_3}\,\mathrm{d}\lambda_1\,]^{-\frac{1}{q_3}}$$

$$=\left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{1j}+b_1)^{\mathrm{m}+\mathrm{a}_1}}{\Gamma\mathrm{m}+\mathrm{a}_1}\int_0^{\infty} \lambda_1{}^{\mathrm{a}_1+\mathrm{m}-q_3-1} e^{-\lambda_1(T_{1j}+b_1)}d\lambda_1\right]^{-\frac{1}{q_3}}$$

$$=\left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{1j}+b_1)^{\mathrm{m}+\mathrm{a}_1}\Gamma\mathrm{m}+\mathrm{a}_1-q_3}{\Gamma\mathrm{m}+\mathrm{a}_1(T_{1j}+b_1)^{\mathrm{m}+\mathrm{a}_1-q_3}}\right]^{-\frac{1}{q_3}}$$

$$=\left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{1j}+b_1)^{q_3}\Gamma\mathrm{m}+\mathrm{a}_1-q_3}{\Gamma\mathrm{m}+\mathrm{a}_1}\right]^{-\frac{1}{q_3}} \qquad (35)$$

Again, minimizing the expectation $E_{\lambda_2}[L_5(\lambda_2, \mathrm{d})]$ and using posterior distribution, we can get the Bayes Estimates of λ_2 using General Entropy Loss Function as shown below:

$$\lambda_{2E}^{*} = [E(\lambda_2{}^{-q_3})]^{-\frac{1}{q_3}}$$

$$= [\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \int_0^{\infty} g\ (\lambda_2|T_{1j}, T_{2j})\ \lambda_2{}^{-q_3}\,\mathrm{d}\lambda_2\,]^{-\frac{1}{q_3}} =$$

$$\left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2}}{\Gamma\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\int_0^{\infty} \lambda_2{}^{\mathrm{a}_2+\mathrm{n}-\mathrm{m}-q_3-1} e^{-\lambda_2(T_{2j}+b_2)}d\lambda_2\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2}}{\Gamma\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\frac{\Gamma\mathrm{n}-\mathrm{m}+\mathrm{a}_2-q_3}{(T_{2j}+b_2)^{\mathrm{n}-\mathrm{m}+\mathrm{a}_2-q_3}}\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{w}} P_j \frac{(T_{2j}+b_2)^{q_3}\Gamma\mathrm{n}-\mathrm{m}+\mathrm{a}_2-q_3}{\Gamma\mathrm{n}-\mathrm{m}+\mathrm{a}_2}\right]^{-\frac{1}{q_3}} \qquad (36)$$

We can obtain the Bayes Estimates $s^*_{1E}(t)$, $s^*_{2E}(t)$, $h^*_{1E}(t)$, $h^*_{2E}(t)$ of $s_1(t)$,$s_2(t)$, $h_1(t)$ and $h_2(t)$ under the General Entropy Loss Function respectively in following manner:

$$s^*_{1E} = [E(s_1{}^{-q_3})]^{-\frac{1}{q_3}}$$

$$= [\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \int_0^\infty s_1{}^{-q_3} g\ (\lambda_1|T_{1j},T_{2j})\lambda_1{}^{-q_3}\,d\lambda_1\,]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{1j}+b_1)^{m+a_1}}{\Gamma m+a_1}\int_0^\infty \lambda_1{}^{a_1+m-1} e^{-\lambda_1(T_{1j}+b_1)} d\lambda_1\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{1j}+b_1)^{m+a_1}}{\Gamma m+a_1}\int_0^\infty e^{\lambda_1 q_3}\,\lambda_1{}^{a_1+m-1} e^{-\lambda_1(T_{1j}+b_1)} d\lambda_1\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{1j}+b_1)^{m+a_1}\Gamma m+a_1}{\Gamma m+a_1(T_{1j}+b_1-q_3)^{m+a_1}}\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \left(1-\frac{q_3}{(T_{1j}+a_2)}\right)^{-(m+a_1)}\right]^{-\frac{1}{q_3}} \qquad (37)$$

Similarly, we can get

$$s^*_{2E} = [E(s_2{}^{-q_3})]^{-\frac{1}{q_3}}$$

$$= [\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \int_0^\infty s_2{}^{-q_3}\, g\left(\lambda_2|T_{1j},T_{2j}\right)\lambda_2{}^{-q_3}\,d\lambda_2\,]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{2j}+b_2)^{n-m+a_2}}{\Gamma n-m+a_2}\int_0^\infty e^{\lambda_2 q_3}\lambda_2{}^{a_2+n-m-1} e^{-\lambda_2(T_{2j}+b_2)} d\lambda_2\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{2j}+b_2)^{n-m+a_2}\Gamma n-m+a_2}{\Gamma n-m+a_2 (T_{2j}+b_2-q_3)^{n-m+a_2}}\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \left(1-\frac{q_3}{(T_{2j}+a_2)}\right)^{-(n-m+a_2)}\right]^{-\frac{1}{q_3}} \quad (38)$$

Here, we have,

$$h_{1E}^{*} = [E(h_1^{-q_3})]^{-\frac{1}{q_3}}$$

$$= [\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \int_0^\infty g\ (\lambda_1 | T_{1j}, T_{2j}) h_1^{-q_3}\, d\lambda_1\ ]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{1j}+b_1)^{m+a_1}}{[\Gamma m+a_1]} \int_0^\infty \lambda_1^{a_1+m-1} e^{-\lambda_1(T_{1j}+b_1)} d\lambda_1\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{1j}+b_1)^{m+a_1}}{[\Gamma m+a_1]} \int_0^\infty \lambda_1^{a_1+m-q_3-1} e^{-\lambda_1(T_{1j}+b_1)} d\lambda_1\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{1j}+b_1)^{m+a_1}[\Gamma m+a_1-q_3]}{[\Gamma m+a_1]\ (T_{1j}+b_1)^{m+a_1-q_3}}\right]^{-\frac{1}{q_3}}$$

$$= \left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{[\Gamma m+a_1-q_3]}{[\Gamma m+a_1]\ (T_{1j}+b_1)^{-q_3}}\right]^{-\frac{1}{q_3}} \quad (39)$$

Similarly, we have

$$h_{2E}^{*} = [E(h_2^{-q_3})]^{-\frac{1}{q_3}}$$

$$= [\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \int_0^\infty g\ (\lambda_2 | T_{1j}, T_{2j})\, h_2^{-q_3}\, d\lambda_1\ ]^{-\frac{1}{q_3}}$$

$$=\left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{2j}+b_2)^{n-m+a_2}}{[\Gamma n-m+a_2]}\int_0^{\infty} \lambda_2^{\,n-m+a_2-1} e^{-\lambda_2(T_{2j}+b_2)} d\lambda_2\right]^{-\frac{1}{q_3}}$$

$$=\left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{2j}+b_2)^{n-m+a_2}}{[\Gamma n-m+a_2]}\int_0^{\infty} \lambda_2^{\,n-m+a_2-q_3-1} e^{-\lambda_2(T_{2j}+b_2)} d\lambda_2\right]^{-\frac{1}{q_3}}$$

$$=\left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{(T_{2j}+b_2)^{n-m+a_2}\,[\Gamma n-m+a_2-q_3]}{[\Gamma n-m+a_2]\,(T_{1j}+b_1)^{n-m+a_2-q_3}}\right]^{-\frac{1}{q_3}}$$

$$=\left[\sum_{m=1}^{n-1}\sum_{j=1}^{w} P_j \frac{[\Gamma n-m+a_2-q_3]}{[\Gamma n-m+a_2]\,(T_{2j}+b_2)^{-q_3}}\right]^{-\frac{1}{q_3}} \qquad (40)$$

5. NUMERICAL STUDY

We have generated 20 random observations from the Two Parameter Rayleigh Model explained earlier. The first ten observations were taken with $\lambda_1 = 0.55$ and the next ten observations were taken with $\lambda_2 = 1.08$ from the same distribution. λ_1 and λ_2 themselves were random observations from gamma distributions with prior means as μ_1=0.55, μ_2=1.08 and coefficient of variations $\Phi_1 = 0.85, \Phi_2 = 0.36$ respectively. The resultant values obtained are given in table 1 for $b_1 = 1.56, b_2 = 2.46, a_1$=1.11 and $a_2 = 3.24$.

TABLE 1

Generated observations from Two Parameter Rayleigh Model

0.2315	0.7731	0.6921	0.1985	0.3170	0.4545	0.2919
0.3927	5.4075	0.2014	0.6870	0.6507	8.2664	0.6894

TABLE 2

Hyper parameter values of the gamma prior and the posterior probabilities

j	**1**	**2**	**3**	**4**
ξ_j	1/13	1/13	1/13	1/13
U_j	0.000198	0.000296	0.000324	1.60639×10^{-8}
b_1	1.08	1.23	1.44	1.53
a_1	1.14	1.16	1.11	1.17
b_2	0.91	1.17	1.62	2.43
a_2	2.07	2.22	2.57	3.24
p_j	0.1757	0.3247	0.4707	0.0261

TABLE 3

Bayes Estimates under Asymmetric Loss Functions

Prior Density	q_1	m_L^*	q_3	m_E^*
Gamma Prior	0.08	9	0.08	9
	0.15	9	0.15	9
	0.26	9	0.26	8
	1.25	8	1.25	7
	1.56	7	1.56	6
	-1.08	8	-1.08	9
	-2.07	9	-2.07	10

TABLE 4 Bayes Estimates under Asymmetric Loss Functions

Prior Density	q_1	λ_{1L}^*	λ_{2L}^*	q_3	λ_{1E}^*	λ_{2E}^*
Gamma prior	0.09	0.52	1.0	0.09	0.53	1.2
	0.10	0.51	1.0	0.10	0.52	1.0
	0.20	0.51	0.9	0.20	0.51	1.0
	1.2	0.48	0.8	1.2	0.49	0.8
	1.5	0.47	0.7	1.5	0.47	0.6
	-1.0	0.54	1.3	-1.0	0.55	1.4
	-2.0	0.56	1.5	-2.0	0.57	1.6

TABLE 5 Bayes Estimates under Asymmetric Loss Functions

Prior Density	q_1	q_3
Gamma Prior	0.09	0.09
	0.10	0.10
	0.20	0.20
	1.2	1.2
	1.5	1.5
	-1.0	-1.0
	-2.0	-2.0

TABLE 6 Bayes Estimates under Asymmetric Loss Functions

Prior Density	q_1	$S_{1L}^*(t)$	$S_{2L}^*(t)$	q_3	$S_{1E}^*(t)$	$S_{2E}^*(t)$
Gamma Prior	0.09	0.28	0.08	0.09	0.30	0.08

	0.10	0.28	0.07	0.10	0.29	0.08
	0.20	0.26	0.06	0.20	0.28	0.08
	1.2	0.25	0.05	1.2	0.28	0.08
	1.5	0.23	0.03	1.5	0.27	0.08
	-1.0	0.31	0.10	-1.0	0.31	0.08
	-2.0	0.33	0.12	-2.0	0.32	0.08

TABLE 7

Bayes Estimates under Asymmetric Loss Functions

Prior Density	q_1	$h_{1L}^*(t)$	$h_{2L}^*(t)$	q_3	$h_{1E}^*(t)$	$h_{2E}^*(t)$
Gamma Prior	0.09	0.48	1.00	0.09	0.49	0.92
	0.10	0.47	0.90	0.10	0.48	0.91
	0.20	0.46	0.90	0.20	0.47	0.90
	1.2	0.45	0.88	1.2	0.47	0.89
	1.5	0.44	0.86	1.5	0.45	0.87
	-1.0	0.50	1.10	-1.0	0.52	1.20
	-2.0	0.52	1.20	-2.0	0.53	1.30

6. CONCLUSIONS

From the numerical studies, we conclude that the Bayes Estimates are robust in nature with respect to the correct choice of the prior specifications on $\lambda_1(\lambda_2)$ and incorrect choice of the prior specifications on λ_2 (λ_1) respectively. It is quite clear that the results are case sensitive in prior specifications on λ_1 and λ_2. Moreover, simultaneous deviations from the true values are clearly seen from the results. Table 3, Table 4, Table 5, Table 6, Table 7 lead to the conclusion that the results are robust with respect to the correct choice of the prior density.

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