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Comparative Analysis of Wavelet Transform and Fourier Transform

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ABSTRACT

This paper analyzes two data analytic methods: Fourier transforms and wavelet transform. Both methods are capable of detecting dominant frequencies in the signals. However wavelet transform is the refinement of Fourier transform. Whereas Fourier transform is only suitable for stationary signal, wavelet is useful for stationary as well as non stationary signal. This paper provides a brief review of both type of transforms and their comparison and shows the superiority of wavelet transform over Fourier transform.

KEYWORDS : Time-frequency signal analysis, Fourier transform, wavelet transform, DWT, CWT.

1. Introduction

During the past five decades, signal processing has grown to become a major discipline in engineering and computer science. Fourier analysis, being the major mathematical tool for signal presentation and processing, breaks down given signal into sinusoidal functions. Since sinusoidal signals are periodic signals, Fourier analysis is an excellent tool for analyzing this class of signals.

The Fourier transform makes use of Fourier series, named in the honour of Joseph Fourier (1770-830). He was the first to use such series to study heat equations. As the Fourier transform decomposes a function of time (signal) into the frequencies that make it up, it is called the frequency domain representation of the original signal. From the original series, various Fourier transforms were derived: the continuous Fourier transform, discrete Fourier transform, Fast Fourier transform, Short- Time Fourier transform etc. Fourier transform is adopted in many applications. But it starts its inefficiency when working the signal with certain characteristics such as aperiodic, noisy, intermittent and transient. In 1946 Dennis Gabor introduced a modification of Fourier transform, namely it Short-Time Fourier Transform [STFT] known later as Gabor transform. Since this Short- Time Fourier transform uses a fixed window function with respect to frequency, both the time as well as frequency resolutions become fixed for all frequencies and times respectively. As a consequence, the short- time Fourier transform does not allow any change in time or frequency resolutions.

Wavelet transform is an approach to the Short-Time Fourier Transform to overcome the resolution problem. Historically, the concept of "Wavelets" originated from the study of time frequency signal analysis. In 1982, Jean Morlet, in collaboration with a group of French engineers, first introduced the idea of wavelets as a family of functions for the analysis of non stationary signals. Wavelets have special ability to examine signal simultaneously in both time and frequency, hence giving time-frequency representation of the signal. Moreover, it does not require the stationarity of the signal. Wavelet transform has overcome the limitations of Fourier transform and therefore, has received more attention. Current applications of the wavelet transform include financial time series analysis, climate analysis, seismic signal denoising, crack surface characterization, audio and video compression, image processing, pattern recognition, fast solution of partial differential equations, computer graphics and so on. P. Addison [9] elaborates the applications of wavelets in all these various fields. Wavelets allow complex information such as music, speech, images and patterns to be decomposed into elementary forms, called the fundamental building blocks, at different positions and scales and subsequently reconstructed with high precision.

This paper focuses on a few aspects of both analytical tools. The second section describes the Fourier analysis from the mathematical point of view and third section discusses wavelet methodology. The fourth section shows similarities and dissimilarities between both types of transforms and last section concludes the paper.

2. Fourier Analysis.

Fourier analysis is useful in many scientific applications. It makes use of Fourier series in dealing with data sets. Joseph Fourier first introduced the remarkable idea of expansion of a function in terms of trigonometric series. In fact Fourier developed his new idea for finding the solution of heat equation in terms of Fourier series so that the Fourier series can be used as a practical tool for determining the Fourier series solution of partial differential equations under prescribed boundary conditions.

Thus the Fourier series [5-6] of a function defined on the interval is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{l}\right)$$

Where the Fourier coefficients are

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(t)\exp\left(\frac{-in\pi t}{l}\right)dt$$

In order to obtain a representation for a non- periodic function defined for all real , it seems to take the limits as that leads to the formulation of the famous Fourier integral theorem

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(iwx)\,dw \int_{-\infty}^{\infty} \exp(-iwt)\,f(t)dt \qquad (1)$$

Mathematically, it is continuous version of completeness property of Fourier series. Physically, this form (1) can be resolved into an infinite number of harmonic components with continuously varying frequency and amplitude

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-iwt) f(t)dt$$

Whereas the ordinary Fourier series represents a resolution of a given function into an infinite discrete set of harmonic components.

The Fourier transform originated from the Fourier integral theorem. Both Fourier series and Fourier transform are related in many important ways. Many applications, including the analysis of stationary signals and real time signal processing, make an effective use of the Fourier transform in the time and frequency domain. The Fourier transform of a signal (function) is defined by

$$F\{f(t)\} = \hat{f}(w) = \int_{-\infty}^{\infty} \exp(\mathrm{iwt}) f(t)\, dt = < f, \exp(iwt) >$$

Where is a function of frequency and is the inner product in a Hilbert space. Thus, the transform of a signal decomposes it into a sine wave of different frequencies and phases, and it is often called the Fourier spectrum. Under certain conditions, the signal can be reconstructed by the Fourier inversion Formula

$$f(t) = F^{-1}\{\hat{f}(w)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-iwt)\, \hat{f}(w) dw = \frac{1}{2\pi} < \hat{f}, \exp(-iwt) >$$

Thus, the Fourier transform theory has been very useful for analyzing harmonic signals or signals for which there is no need for local information. Also it is very useful in areas such as quantum mechanics, wave motion and turbulence.

In spite of some remarkable success, Fourier transform analysis seems to be inadequate for studying some physical problems for at least two reasons. First, the Fourier transform of a signal does not contain any local information. Second, it enables us to investigate the problems either in the time (space) domain or in the frequency domain, but not simultaneously in both domains. These are probably the major weaknesses of the Fourier transform analysis.

2. Wavelet Analysis
Wavelet analysis is a refinement of Fourier analysis. Wavelet transform overcomes most of the limitations of the Fourier transform. It gives local information of the signal. Wavelet transform allows us to study the frequency components of the signal with time information simultaneously. So it is very useful for non stationary signal analysis. Firstly, Alfred Haar used the term 'wavelet' in his thesis in 1909. In 1982, J. Morlet first introduced the idea of wavelet transform as a new mathematical tool for seismic signal analysis. A. Grossman developed an exact inversion formula for the wavelet transform. Y. Meyer discovered a new kind of wavelet, with a mathematical property called orthogonality that made the wavelet transform an easy to work with and manipulate as a Fourier transform. Stephane Mallat constructed wavelet decomposition and reconstruction algorithms using Multiresolution Analysis (MRA), which is known as the heart of wavelet theory. The major achievement of wavelet analysis is due to Daubechies who discovered a whole new class of wavelets which are non orthogonal and could be implemented using short digital filters. Her work had a tremendous positive impact on the study of wavelets and their applications.

Wavelets mean 'small waves'. They have finite length and oscillatory behaviour. Like Fourier transform, wavelet transform deals with expansions of functions in terms of set of basis functions. But wavelet transform [10] expands the function in the form of translation and dilation of a fixed function. The wavelets obtained in this way have special scaling properties. They permit a closer connection between the function (signal) being represented and their coefficients. There are two types of wavelet transform- the continuous wavelet transform (CWT) and its discretized version, discrete wavelet transform (DWT). The CWT is popular among physicists, whereas the DWT is more common in numerical analysis, signal and image processing. The DWT produces only the minimal number of coefficients necessary to reconstruct the original signal. In the DWT the number of observations has to be dyadic i.e.an integer power of 2. There is variety of wavelets such as Daubechies, Symlet, Meyer, Morlet, etc and the choice of the mother wavelets depends on the characteristics of data.

Wavelet consists of the dilations and translations of a single real valued function, called mother wavelet [4]. This function must have unit energy, zero average and also satisfies admissibility condition. Dilation of function corresponds to either a spreading out or contraction of the function and translation means a shift of the argument along the real axis. So for any mother wavelet, we define a family of functions by the dilation and translation of ,

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right); a, b \in \mathbb{R}, a > 0$$

Each is called a wavelet. Here the factor is introduced for normalization necessary to have an orthonormal wavelet basis.

We may represent any function f∈L^2 (R) by

$$(Wf)(a,b) = < f, \psi_{a,b} > = \int_{-\infty}^{\infty} f(t) \psi_{a,b}(t) dt$$

Wf is called the continuous wavelet transform of . By combining several combination of dilation and translation of the mother wavelet, the wavelet transform is able to capture all information in the signal and associate it with specific time horizons and locations in time.

Given $a_0 > 1$ and $b_0 \neq 0$, we may restrict a and b respectively, to the discrete lattices $a \in \{a_0^m : m \in \mathbb{Z}\}$, $b \in \{n a_0^m b_0 : m, n \in \mathbb{Z}\}$. Then $\psi_{a,b}$ becomes

$$\psi_{m,n} = a_0^{\frac{-m}{2}} \psi(a_0^{-m} t - n b_0)$$

And Wf becomes

$$(Wf)(m,n) = < f, \psi_{m,n} > = \int_{-\infty}^{\infty} f(t) \psi_{m,n}(t) dt \qquad (2)$$

The choice of b_0 is arbitrary and by convention is taken to be 1. On the other hand, the choice of significantly affects the properties of . For multiresolution analysis, we want to be orthonormal. Taking allows us to define ψ such that the are orthonormal. This is also the conventional choice for . Then the form an orthonormal bases of . With this choice, equation (2) is called dyadic wavelet transform.

Many wavelets have rapid decay. Meyer constructed a wavelet that decays faster than any power. Lemarie and Battle independently constructed a collection of wavelets that decay exponentially. Orthogonal wavelets may be classified as either having compact support or non compact support. I. Daubechies [3] characterized all orthogonal wavelets with compact support. She showed that compactly supported wavelets may be chosen with arbitrary regularity; however, the support width varies directly with the regularity. The compact support of Daubechies wavelets and the rapid decay of the wavelets described by Meyer and Lemarie and Battle help provide for both the time-localization ability and efficient computation of the wavelet transform.

3. Comparison of Wavelet analysis with Fourier analysis
In this section we make a comparison of Wavelet transform with Fourier transform and see that wavelet transform is better than Fourier transform.

- A signal is called stationary if there is no change in the properties of signal and vice versa for non-stationary signal. Fourier transform is a powerful tool for analyzing the components of a stationary signal. But it is less useful in analyzing non-stationary signal. Wavelet transform allows the components of non-stationary signal to be analyzed. Wavelets also allow filters to be constructed for stationary and non-stationary signals.
- Both Fourier transform and wavelet transform are given by integral equations in the form of a correlation. In the Fourier transform the correlation is with dilations of the function and in the wavelet transform the correlation is with dilations and translations of the mother wavelet ψ, which can be any wavelet. Also both transforms are invertible and linear operator.
- Both transforms may take real or complex functions as their input. The output of the Fourier transform is always complex.

However, there are both real and complex-valued wavelets. If a real-valued mother wavelet is used, the wavelet transform may be real or complex-valued (real-valued if the input function is real-valued and complex-valued if the input function is complex-valued). If a complex-valued wavelet is used as the mother wavelet, the wavelet transform is complex-valued.

- Fourier transforms are great, but they capture global features of signal and so local features can get lost. On the other side wavelet transform analyses the signal locally.
- Fourier analysis transforms the time-signal to frequency-based signal. Here, transforming to the frequency domain, time information is lost, we don't know when an event occurred. Wavelet analysis transforms the signal to time and frequency-based signal. Fourier transform tells us what frequencies are present in signal, but wavelet transform tells us what frequencies exist and where (at what time (scale)). In the other words we can say that Fourier transform is only localized in frequency domain but wavelet transform is well localized in both time and frequency domain. This localization feature makes many functions and operators using wavelets "sparse" when transformed into the wavelet domain. This sparseness, in turn, results in a number of useful applications such as data compression, detecting features in images, and removing noise from time series.
- Wavelet transforms do not have a single set of basis functions like the Fourier transform, which utilizes just sine and cosine functions. Instead, wavelet transforms have infinite set of possible basis functions. Thus wavelet analysis provides immediate access to information that can be obscured by Fourier analysis. The mathematics of wavelets is much larger than that of Fourier transform. In fact the mathematics of wavelets encompasses the Fourier transform.
- The short- time Fourier transform (STFT), a version of Fourier transform, uses a fixed window function with respect to frequency. The original signal is partitioned into small enough segments such that these portions of the non stationary signal can be assumed to be stationary over the duration of the window function. Once the window function is chosen, it remains fixed for all frequencies and times respectively. As a consequence, the short- time Fourier transform does not allow any change in time or frequency resolutions. On the other side, wavelet transform analyzes the signal at different resolutions using multiresolution analysis. The multiresolution analysis approach may overcome the resolution problem as it adaptively partitions the time frequency plane, using short windows at high frequencies and long windows at low frequencies and thus letting both time and frequency resolutions to vary in the time –frequency plane. Wavelets often give a better signal representation using multiresolution analysis.
- Wavelet analysis is capable of revealing aspects of signal (like trends, breakdown points, and discontinuities in higher derivatives and self-similarity) that Fourier analysis misses. Also using fast wavelet transform, wavelets are computationally very fast.

M. Akin [7] analyzed EEG signals using Fast Fourier Transform (FFT) and wavelet methods and found that wavelet transform method is better in detecting brain diseases. In [8], M. Sifuzzaman discussed some applications of wavelets such as data compression, music signal, and fingerprint verification and also did a comparison of Fourier and wavelet transform. A. Drozdov [1] showed that wavelet transform can be used as a higher quality method for finding quasi-harmonic components in any signals that have non-stationary behaviour. J. Houtveen [4] also proved that both Fourier and wavelet methods perform almost identically for computation of heart period power values but wavelet methods are only superior for analyzing heart period data when additional analyses in the time-frequency domain are required.

Conclusion

In the present paper, comparison of two different signal analysis tools has been discussed. The developments in wavelet analysis can be viewed as resolving some of the difficulties inherent in Fourier analysis. Any application using the Fourier transform can be formulated by wavelet transform to provide more accurately localized time and frequency information. Due to its advantages over Fourier analysis, wavelet analysis has attracted much attention recently in many fields such as electrical engineering (signal processing, data compression), mathematical analysis (harmonic analysis, operator theory) and physics (fractal, quantum theory). Finally, we can say that for signal analysis, wavelet analysis is more suitable and reliable than Fourier analysis.

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