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Mellin Transformation for Extreme Values of Eigenvalues and Upper Percentile Points

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ABSTRACT

In this paper, we define the Mellin Transformation, then use it to derive the joint distribution of two eigenvalues. This form of distribution was first defined by Roy(1939), Hsu(1939), and Fisher(1939). We also tabulate the upper percentile points of this distribution that was partially done by Pillai K.C.S. (1956).

KEYWORDS : Eigenvalues, Extended Tables, Fisher-Girshick-Hsu-Roy Distribution, Upper Percentage points, Pillai K.C.S.

1.INTRODUCTION

Pillai K.C.S.(1956) has defined the cumulative distribution function of the largest root of eigenvalues that involves only two incomplete beta functions. Later, he neglects some higher order terms to approximate upper percentile points of 5% and 1%. This well-known distribution depends on the number of characteristic roots and two parameters m and n. They are defined differently for various situations as described by Pillai (1955). The upper percentage points of the distribution are commonly used in three different types of hypothesis testing in multivariate analysis : (1) test of equality of the variance-covariance matrices of two p-variate normal population, (2) test of equality of the p-dimensional mean vectors for k p-variate normal populations, and (3) test of independence between a p-set and a q-set of variates in a (p+q)-variate normal population. When the null hypotheses to be tested are true, all the three types of test proposed above have been shown to depend only on the characteristic roots of matrices using observed samples. Using a random sample from the multivariate normal population, we could compute the characteristic roots from a usual sum of product matrices of this sample. We then compare the largest characteristic root of the matrices with the percentage points that we have tabulated in this paper to determine whether or not to reject the null hypothesis at a certain probability confidence. For this reason, the percentage points of the largest characteristic roots of the distribution have deeply attracted the attention of mathematical statisticians more than six decades. In this paper, we first define the Mellin Transformation, then use it to find the cumulative distribution function of the largest eigenvalues distribution. We found that our results consistent with Pillai K.C.S. We have carefully defined our m and n values for different tests purposes. Our selected m and n values are much wider than Pillai, however, at the duplicated case results are consistent**.** The included upper percentage points are 0.80,0.82 0.825, 0.850, 0.875, 0.890, 0.900, 0.910(0.005), 0.995. Different authors have selected different m and n parameter values. In our tables, the two parameters are selected in such a way that all existing table values will include. For the parameter m=0(1)15 and the parameter n=1(1)20(2)30(5)80(10)150, 200(100) 1000, Chen W.W.S.(2003) (2004) has established some similar tables for the cases when s=5, and s=7. We only include a tiny portion of the tables in this paper. For interested readers can contact author for the detailed completed tables. In the section 2, we define the Mellin Transformation and then use it to derive the extreme eigenvalue distribution. In the section 3, we give some useful code that has been used to compute the percentile points of this paper.

2. Mellin Transformation

$$\text{Let } J_{m,n,x}(g(t)) = \int_0^x t^m(1-t)^n g(t)dt \text{ then}$$

$$J_{m+1,n,x}(g) = \frac{m+1}{m+n+2} J_{m,n,x}(g) + \frac{J_{m+1,n+1,x}(g')}{m+n+2} - \frac{1}{m+n+2} x^{m+1}(1-x)^{n+1} g(x)$$

$$\text{Proof: Since } J_{m,n,x}(g) = \int_0^x t^m(1-t)^n g(t)dt$$

$$\text{Let } u = (1\text{-}t)^{m+n+2} g(t) \quad dv = \frac{1}{(1\text{-}t)^2}\left(\frac{t}{1-t}\right)^m dt$$

$$du = -(m+n+2)(1\text{-}t)^{m+n+1} g(t)dt + (1\text{-}t)^{m+n+2} g'(t)dt$$

$$v = \left(\frac{1}{m+1}\right)\left(\frac{t}{1\text{-}t}\right)^{m+1} \text{ then}$$

$$J_{m,n,x}(g) = \frac{1}{m+1} x^{m+1}(1\text{-}x)^{n+1} g(x) + \frac{m+n+2}{m+1}\int_0^x t^{m+1}(1-t)^n g(t)dt$$

$$-\frac{1}{m+1}\int_0^x t^{m+1}(1-t)^{n+1} g'(t)dt$$

$$= \frac{1}{m+1} x^{m+1}(1\text{-}x)^{n+1} g(x) + \frac{m+n+2}{m+1} J_{m+1,n,x}(g) - \frac{1}{m+1} J_{m+1,n+1,x}(g')$$

$$J_{m+1,n,x}(g) = \frac{m+1}{m+n+2} J_{m,n,x}(g) + \frac{1}{m+n+2} J_{m+1,n+1,x}(g') - \frac{x^{m+1}(1-x)^{n+1} g(x)}{m+n+2}$$

Derivation of extreme distribution

$$I = \int_0^x \int_{x_1}^x \theta_1^m \theta_2^m (1-\theta_1)^n (1-\theta_2)^n (\theta_1 - \theta_2) d\theta_1 d\theta_2$$

$$\text{Let } g_{m,n,x} = \int_{x_1}^x \theta_2^m (1-\theta_2)^n d\theta_2 = \int_0^x u^m (1-u)^n du \text{ and}$$

$$J_{m,n,x} = \int_0^x \theta_1^m (1-\theta_1)^n g_{m,n,x} d\theta_1$$

$$I = \int_0^x \int_{x_1}^x \theta_1^{m+1} \theta_2^m (1-\theta_1)^n (1-\theta_2)^n d\theta_1 d\theta_2 - \int_0^x \int_{x_1}^x \theta_1^m \theta_2^{m+1} (1-\theta_1)^n (1-\theta_2)^n d\theta_1 d\theta_2$$

$$\text{aware the fact : } \int_{x_1}^x = \int_0^x - \int_0^{x_1} = \int_0^x + \int_{x_1}^0$$

$$I = J_{m+1,n,x}(g_{m,n}) - \int_0^x \int_{x_1}^x \theta_1^m \theta_2^{m+1} (1-\theta_1)^n (1-\theta_2)^n d\theta_1 d\theta_2$$

$$= J_{m+1,n,x}(g_{m,n}) - \int_0^x \theta_2^{m+1} (1-\theta_2)^n d\theta_2 \int_{x_1}^x \theta_1^m (1-\theta_1)^n d\theta_1$$

$$= J_{m+1,n,x}(g_{m,n}) - \int_0^x \theta_2^{m+1}(1-\theta_2)^n d\theta_2 \{\int_0^x \theta_1^m (1-\theta_1)^n d\theta_1 - \int_0^{\hat{1}} \theta_1^m (1-\theta_1)^n d\theta_1\}$$

$$= J_{m+1,n,x}(g_{m,n}) - \int_0^x \theta_2^{m+1}(1-\theta_2)^n d\theta_2 \int_0^x \theta_1^m (1-\theta_1)^n d\theta_1 + J_{m+1,n,x}(g_{m,n})$$

$$I = 2J_{m+1,n,x}(g_{m,n}) - g_{m+1,n} g_{m,n}$$

$$J_{m+1,n,x}(g_{m,n}) = \frac{m+1}{m+n+2} J_{m,n,x}(g) + \frac{1}{m+n+2} \int_0^x t^{m+1}(1-t)^{n+1} t^m (1-t)^n dt$$

$$- \frac{1}{m+n+2} x^{m+1}(1-x)^{n+1} g_{m,n}$$

$$= \frac{m+1}{m+n+2} J_{m,n,x}(g) + \frac{1}{m+n+2} J_{m+1,n+1,x}(g') - \frac{x^{m+1}(1-x)^{n+1} g_{m,n}}{m+n+2}$$

$$J_{m,n}(g_{m,n}) = \int_0^x t^m (1-t)^n \int_0^x u^m (1-u)^n du dt$$

Since $\int_0^x g(t) \int_0^x g(u) du dt = \frac{1}{2}[\int_0^x g(t) dt]^2$ $\quad J_{m,n}(g_{m,n}) = \frac{1}{2}\{g_{m,n}\}^2$

Second term in $J_{m+1,n+1,x}(g'_{m,n})$ is $\frac{1}{m+n+2} g_{2m+1,2n+1}$

where $g_{m,n} = \{\int_0^x t^m (1-t)^n dt\}$

$$I = \{g_{m,n}\}^2 \frac{m+1}{m+n+2} + \frac{2}{m+n+2} g_{2m+1,2n+1} - \frac{2}{m+n+2} x^{m+1}(1-x)^{n+1} g_{m,n}$$

$$-(g_{m+1,n} g_{m,n})$$

$$= \frac{m+1}{m+n+2} \{\int_0^x t^m (1-t)^n dt\}^2 + \frac{2}{m+n+2} \{\int_0^x t^{2m+1}(1-t)^{2n+1} dt\}$$

$$- \frac{2x^{m+1}(1-x)^{n+1}}{m+n+2} \{\int_0^x t^m (1-t)^n dt\} - \{\int_0^x t^{m+1}(1-t)^n dt\}\{\int_0^x t^m (1-t)^n dt\}$$

Check the coefficient of incomplete beta function;

Second term in k(2,m,n)*I is

$$\frac{1}{4(m+n+2)} \frac{\Gamma(2m+2n+5)}{\Gamma(2m+2)\ \Gamma(2n+2)} \frac{\Gamma(m+1)\ \Gamma(n+1)}{\Gamma(m+n+2)}$$

$$= \frac{1}{4(m+n+2)} \frac{(2m+2n+4)!}{(2m+1)!(2n+1)!} \frac{m!n!}{(m+n+1)!}$$

$$= \frac{1}{4} \frac{(2m+2n+4)!}{(2m+1)!(2n+1)!} \frac{m!n!}{(m+n+2)!} \frac{(2m+2)(2n+2)}{(2m+2)(2n+2)}$$

$$= \frac{(2m+2n+4)!}{(2m+2)!(2n+2)!} \frac{(m+1)!(n+1)!}{(m+n+2)!}$$

$$=\frac{\binom{2m+2n+4}{2m+2}}{\binom{m+n+2}{m+1}}$$

For $s=2$ $pr\{x<\theta\}$

$$=\mathrm{Beta}(\theta,\ 2m+2,2n+2)-\frac{\binom{2m+2n+4}{2m+2}}{\binom{m+n+2}{m+1}}x^{m+1}(1-x)^{n+1}\mathrm{Beta}(\theta,m+1,n+1)$$

where $\mathrm{Beta}(\theta,\ a,b)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^\theta t^{a-1}(1-t)^{b-1}dt$

Apply above transformation in Generalized beta

For $n_h<p$, change (n_h and p) to (n_e and n_e-p+n_h) Following Roy,

Let exponent $(n_h-p-1)/2$ called m; $(n_e-p-1)/2$ called n

$$\prod_{i=1}^{p}\Gamma(\frac{n_e-i+n_h+1}{2})=\prod_{i=1}^{p}\Gamma(\frac{n_e-p+n_h+i}{2})\qquad \text{where i(new) = p-i+1(old)}$$

$$=\prod_{i=1}^{p}\Gamma(\frac{n_h-p-1}{2}+\frac{n_e-p-1}{2}+\frac{p}{2}+\frac{i}{2}+1)=\prod_{i=1}^{p}\Gamma(m+n+\frac{p+i+2}{2})$$

Also $\prod_{i=1}^{p}\Gamma(\frac{p-i+1}{2})(\text{old})=\prod_{i=1}^{p}\Gamma(\frac{i}{2})(\text{new})$

$$\prod_{i=1}^{p}\Gamma(\frac{n_h-i+1}{2})=\prod_{i=1}^{p}\Gamma(\frac{n_h-(p-i+1)+1}{2})=\prod_{i=1}^{p}\Gamma(\frac{n_h-p+i}{2})$$

$$=\prod_{i=1}^{p}\Gamma(\frac{n_h-p-1}{2}+\frac{i+1}{2})=\prod_{i=1}^{p}\Gamma(m+\frac{i+1}{2})$$

Roy's standard for the distribution of largest root , let

$s=\min\{n_h,p\}$ (dimension); $m=\frac{|n_h-p|-1}{2}$ (exponent of x_i);

$n=\frac{n_e-p-1}{2}$ (exponent of 1-x_i);

3.SOME USEFUL CODE

From previous section we observed that the probability level of eigenvalues is the mixed sum of two incompleted Beta function. Hence evaluating the incompleted Beta function turns out to be the key to compute our percentile points. However Beta function related to compute the well known gamma function. It is an immediate requirement how to evaluate the gamma function. Most beginner will use the factorial method to program this function. Then he will be caught in the trap. For example, if we wish to compute the gamma(46)=45!, this value already exceeds the upper limit of most current computer. The computer will respond an error message : “overflow.” The correct way should code in ln(gamma) function and then take exponential when we need gamma value. The Abramowitz. M and Stegum. Handbook of mathematics function has a good source for this function. The following available code is the author used to compute the required percentile points.

```
double precision function dlggm(dx)
implicit real*8(a-h,o-z)
real*8 rn45,dx,dy,dterm,da,db,domeg,ds,dz,dw,dz15,
+ dv,du,dt,dr,dq,dp,rn0,rn1,rn2i,eps,dyi
data dz15/-0.29550653594771241830065359477d-1/,rn45/45.d0/
data rn0,rn1,rn2i,da,db,domeg,eps,dz,dw,dv,du,dt,dr,dq,dp,
in2/
+  0.d0,1.d0,0.5d0,0.9999999999999999999999999999d0,
+  1.0000000000000000000000000001d0, 1.d35, 0.10d-25,
+  0.641025641025641025641025641 03d-2,
+ -1.917526917526917526917526917 3d-3,
+  0.841750841750841750841750841 75d-3,
+ -5.952380952380952380952380952 4d-4,
+  7.936507936507936507936507936 50d-4,
+ -2.777777777777777777777777777 77d-3,
+  8.333333333333333333333333333 33d-2,
+  0.91893853320467274178032973643d0, 2/
dy=dx
dterm=rn1
c
c***** check the limitation
c
dlggm=-domeg
if(dx.le.eps) go to 98
c
c*** if dx is within 0.1d-28 of 1.0 or 2.0 we set dlggm to 0
c
dlggm=rn0
if(dx.ge.da.and.dx.lt.db) go to 99
if(dx.ge.da+rn1.and.dx.lt.db+rn1) go to 99
c
c**** to ensure the result with greater than 0.1d-27
c**** the argument dx must be greater than 45
c
2     if(dy.gt.rn45)go to 4
dterm=dterm*dy
dy=dy+rn1
go to 2
```

```
4     dyi=rn1/dy
ds=dyi*dyi
c*** calculate the result using the first eight terms of the
c*** series
dlggm=(dy-rn2i)*dlog(dy)+dp-dy-dlog(dterm)+
+          (((((((dz15*ds+dz)*ds+dw)*ds+dv)*ds+du)*ds+dt)
*ds+dr)*ds+dq)*dyi
go to 99
98    write(6,101)dx
101   format(//,'error the argument dx is less than 0.1d-
25',f30.29)
c
99    return
end
```

$$B(x;\alpha,\beta)=\frac{x^{\alpha}(1-x)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta)}\left[\frac{1}{1+}\frac{C_1}{1+}\frac{C_2}{1+}\frac{C_3}{1+}\ldots\ldots\ldots\frac{C_n}{1+}\ldots\ldots\ldots\right]$$

where $$C_{2k}=\frac{-(\alpha+k-1)(\alpha+\beta+k-1)}{(\alpha+2k-2)(\alpha+2k-1)}x,$$

and $$C_{2k+1}=\frac{k(\beta-k)}{(\alpha+2k-1)(\alpha+2k)}x, \qquad k=1,2,3,\ldots\ldots$$

This formula is most efficiently used when x is less than its Mean (i.e. $x<\mu_x=\frac{\alpha}{\alpha+\beta}$), but when $x>\mu_x$, we may simply take complements (i.e. $B(x;\alpha,\beta)=1-B(1-x;\beta,\alpha)$)

The computer program module called BETAC evaluates the Incomplete Beta function using the continued fraction as defined in (1). This computer code lists below.

```
Function BETAC(x,a,b)
Common rm
apb=a+b
alo=rm=0.0
blo=bhi=bev=bod=1.0
ahi=exp(dlggm(apb)+a*alog(x)+b*alog(1.0-x)-dlggm(a+1.0)-
+    dlggm(b))
f=fx=ahi
10   rm=rm+1.0
rm1=rm-1.0
rev=-(a+rm1)*(a+b+rm1)*x/((a+rm1+rm1)*(a+rm+rm1))
aod=rm*(b-rm)*x/((a+rm+rm1)*(a+rm+rm))
alo=bev*ahi+aev*alo
blo=bev*bhi+aev*blo
ahi=bod*alo+aod*ahi
bhi=bod*blo+aod*bhi
if(bhi.eq.0.0) go to 10
f=ahi/bhi
if(abs((f-fx)/f).lt.1.0e-10) go to 20
fx=f
go to 10
20   betac=f
return
end
```

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