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On the Convergence of A Newly Developed Non-Linear Computational Algorithm

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ABSTRACT *This paper presents a comprehensive detail of the convergence theorem for a newly developed non-linear method for numerical integration of ordinary differential equations. It is proved that the scheme is convergent.*

KEYWORDS :Convergence, Numerical integration, ordinary differential equation, computational Algorithm

INTRODUCTION 1.0

We have established the numerical integration algorithm (1.1) which can be expressed as one-step methods in the form:

$$y_{n+1} = y_n + \left[\frac{f_n^3 - \lambda f_n^2 + f_n^1 - \lambda f_n}{2(1-\lambda x_n)}\right](2n+1)h^2 + \left[\frac{f_n^2 cosx_n - f_n^3 sinx_n}{\lambda^3(cosx_n - \lambda sinx_n)}\right]\left[\sum_{r=1}^{\infty}\frac{(\lambda h)^r}{r!}\right]$$

$$+\left[\frac{f_n^3 - \lambda f_n^2}{cosx_n - \lambda sinx_n}\right]\left[\cos x_n\left(\sum_{r=1}^{\infty}\frac{h^{2r}}{2r!}\right) - \sin x_n\left(\sum_{r=0}^{\infty}\frac{h^{2r+1}}{(2r+1)!}\right)\right] \quad 1.1$$

$$y_{n+1} = y_n + h\emptyset(x_n, y_n; h)$$

Where

$$\emptyset(x_n, y_n; h) = \{Pf_n + Qf_n^1 + Rf_n^2 + Sf_n^3\} \quad 1.2$$

is the increment function.

Definition 1.0 (Henrici, 1962)

We define any algorithm for solving a differential equation in which the approximatic the solution at the point x_{n+1} can be calculated if only x_n, y_n and h are known as method. It is a common practice to write the functional dependence, y_{n+1}, on the quanti and h in the form $y_{n+1} = y_n + h\emptyset(x_n, y_n; h)$.

We observe that $\emptyset(x_n, y_n; h) = \{Pf_n + Qf_n^1 + Rf_n^2 + Sf_n^3\}$ 1.3

2.0 CONVERGENCE OF THE NON- LINEAR COMPUTATIONAL ALGORIT

Theorem 1.0 (Henrici, 1962)

$x \in [a,b]$ *and* $y \in (-\infty,\infty)$; $0 \le h \le h_0$, where $h_0 > 0$, and let there exists a constan that $|\emptyset(x_n, y_n^*; h) - \emptyset(x_n, y_n; h)| \le L|y_n^* - y_n|$

For all $(x_n, y_n; h)$ and $(x_n, y_n^*; h)$ in the region just define.

Then the relation $(x_n, y_n; 0) = (x_n, y_n)$ is a necessary and sufficient condition convergence of the new scheme defined above in (1.3)

Proof

The increment function $\emptyset(x_n, y_n; h)$ can be written in the form:

$$\emptyset(x_n, y_n; h) = \{Pf_n + Qf_n^1 + Rf_n^2 + Sf_n^3\} \qquad 2.0$$

Where P, Q, R and S are coefficients defined below:

$$P = -\frac{\lambda(2n+1)}{2(1-\lambda x_n)}h$$

$$Q = \frac{h(2n+1)}{2(1-\lambda x_n)}$$

$$R = \frac{\lambda(2n+1)h}{2(1-\lambda x_n)} + \left(\frac{12 + 6h + 2h^2\lambda^2}{12\lambda^2} - \frac{\lambda h^7 - 56\lambda h^5 - 20160h\lambda}{40320}\right)\frac{cosx_n}{cosx_n - \lambda sinx_n}$$
$$+\left(\frac{\lambda h^6 - 42\lambda h^4 + 840\lambda h^2 - 210\lambda h^3}{5040}\right)\frac{sinx_n}{cosx_n - \lambda sinx_n}$$

$$S = \frac{2n+1}{2(1-\lambda x_n)} + \left(\frac{(2-h\lambda)}{2\lambda^2} - \frac{210\lambda h^3 - 5040 - 42h^4 - h^6}{5040}\right)\frac{sinx_n}{cosx_n - \lambda sinx_n}$$
$$+\left(\frac{20160h + 1680h^3 + 56h^5 + h^7}{40320}\right)\frac{cosx_n}{cosx_n - \lambda sinx_n}$$

Equation (2.2), can also be written as,

$$\emptyset(x_n, y_n^*; h) = \left\{Pf(x, y_n^*) + Qf^1(x, y_n^*) + Rf^2(x, y_n^*) + Sf^3(x, y_n^*)\right\} \qquad 2.1$$

Subtract (2.0) from (2.1), hence:

$$\emptyset(x_n, y_n^*; h) - \emptyset(x_n, y_n; h) = \left\{ \begin{matrix} P\big(f(x, y_n^*) - f(x, y_n)\big) + Q\left(f^1(x, y_n^*) - f^1(x, y_n)\right) \\ +R(f^2(x, y_n{}^*) - f^2(x, y_n)) + S\left(f^3(x, y_n{}^*) - f^3(x, y_n)\right) \end{matrix} \right\}$$

Let $\bar{y}$ be defined as a point in the interior of the interval whose points are y and y^*, applying mean value theorem, we have:

$$\left. \begin{aligned} f(x_n, y_n^*) - f(x_n, y_n) &= \frac{\partial f(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n) \\ f'(x_n, y_n^*) - f'(x_n, y_n) &= \frac{\partial f'(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n) \\ f^2(x_n, y_n^*) - f^2(x_n, y_n) &= \frac{\partial f^2(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n) \\ f^3(x_n, y_n^*) - f^3(x_n, y_n) &= \frac{\partial f^3(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n) \end{aligned} \right\} \qquad 2.2$$

$$\left. \begin{aligned} L &= \sup_{(x_n, y_n) \in D} \frac{\partial f(x_n, y_n)}{\partial y_n} \\ L_1 &= \sup_{(x_n, y_n) \in D} \frac{\partial f'(x_n, y_n)}{\partial y_n} \\ L_2 &= \sup_{(x_n, y_n) \in D} \frac{\partial f^2(x_n, y_n)}{\partial y_n} \\ L_3 &= \sup_{(x_n, y_n) \in D} \frac{\partial f^3(x_n, y_n)}{\partial y_n} \end{aligned} \right\} \qquad 2.3$$

Therefore

$$\emptyset(x_n, y_n^*; h) - \emptyset(x_n, y_n; h) = P\left\{\frac{\partial f(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n)\right\} + Q\left\{\frac{\partial f'(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n)\right\}$$

$$+R\left\{\frac{\partial f^2(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n)\right\} + S\left\{\frac{\partial f^3(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n)\right\}$$

$$= PL(y_n^* - y_n) + Q\,L_1(y_n^* - y_n) + R\,L_2(y_n^* - y_n) + S\,L_3(y_n^* - y_n) \qquad 2.4$$

Taking the absolute value of both sides

$$|\emptyset(x_n, y_n^*; h) - \emptyset(x_n, y_n; h)| \leq |PL + Q\,L_1 + R\,L_2 + S\,L_3||y_n^* - y_n| \qquad 2.5$$

If we let $M = |PL + Q\,L_1 + R\,L_2 + S\,L_3|$ then, our equation (2.5) becomes

$$|\emptyset(x_n, y_n^*; h) - \emptyset(x_n, y_n; h)| \leq M|y_n^* - y_n| \qquad 2.6$$

which is the condition for convergence.

3.0 CONSISTENCE OF THE NON-LINEAR COMPUTATIONAL ALGORITHM

Definition 2.0 (Fatunla, 1988)

A numerical scheme with an increment function $\emptyset(x_n, y_n; h)$ is said to

be consistent with the initial value problem

$$y' = f(x,y); \qquad y(x_0) = y_0 \qquad 3.0$$

From (1.2), $y_{n+1} = y_n + h\emptyset(x_n, y_n; h)$

Where

$$\emptyset(x_n, y_n; h) = \{Pf_n + Qf_n^1 + Rf_n^2 + Sf_n^3\} \qquad 3.1$$

If $h = 0$, then

$$y_{n+1} = y_n + \left[\frac{f_n^3 - \lambda f_n^2 + f_n^1 - \lambda f_n}{2(1-\lambda x_n)}\right](2n+1)h^2 + \left[\frac{f_n^2 cosx_n - f_n^3 sinx_n}{\lambda^3(cosx_n - \lambda sinx_n)}\right]\left[\sum_{r=1}^{\infty}\frac{(\lambda h)^r}{r!}\right]$$

$$+ \left[\frac{f_n^3 - \lambda f_n^2}{cosx_n - \lambda sinx_n}\right]\left[\cos x_n\left(\sum_{r=1}^{\infty}\frac{h^{2r}}{2r!}\right) - \sin x_n\left(\sum_{r=0}^{\infty}\frac{h^{2r+1}}{(2r+1)!}\right)\right]$$

Becomes

$y_{n+1} = y_n$, hence

$$\emptyset(x_n, y_n; 0) = f(x,y) \qquad 3.2$$

Therefore a consistent method has order of at least one. We say our new numerical metho consistence since equation (2.5) reduces to (3.2) when $h = 0$, therefore, we say that the methc consistent

4.0 STABILITY ANALYSIS OF THE NON-LINEAR COMPUTATIONAL ALGORITHM

THEOREM 2.0 (Fatunla, 1988)

Let $y_n = y(x_n)$ and $p_n = p(x_n)$ denote two different numerical solution of differential equation (3.0) with the initial conditions specified as

$y(x_0) = \zeta$ and $p(x_0) = \zeta^*$ respectively, such that $|\zeta - \zeta^*| < \varepsilon, \quad \varepsilon > 0$.

If the two numerical estimates are generated by the integration scheme, we have:

$$y_{n+1} = y_n + h\emptyset(x_n, y_n; h) \qquad 4.0$$

$$p_{n+1} = p_n + h\emptyset(x_n, p_n; h) \qquad 4.1$$

The condition that

$$|y_{n+1} - p_{n+1}| \leq k|\zeta - \zeta^*|$$

Is the necessary and sufficient condition that our new method is stable and convergent.

Proof:

If $y_{n+1} = y_n + h\{Pf_n + Qf_n^1 + Rf_n^2 + Sf_n^3\}$ 4.2

Then let

$$y_{n+1} = y_n + h\left\{Pf(x_n, y_n) + Qf^1(x_n, y_n) + Rf^2(x_n, y_n) + Sf^3(x_n, y_n)\right\} \qquad 4.3$$

And

$$p_{n+1} = p_n + h\left\{Pf(x_n, p_n) + Qf^1(x_n, p_n) + Rf^2(x_n, p_n) + Sf^3(x_n, p_n)\right\} \qquad 4.4$$

Therefore

$$y_{n+1} - p_{n+1} = y_n - p_n + h \left\{ \begin{array}{c} P\big(f(x_n, y_n) - f(x_n, p_n)\big) \\ +Q\left(f^1(x_n, y_n) - f^1(x_n, p_n)\right) \\ +R\left(f^2(x_n, y_n) - f^2(x_n, p_n)\right) \\ +S\left(f^3(x_n, y_n) - f^3(x_n, p_n)\right) \end{array} \right\} \quad 4.5$$

Applying the mean value theorem, we have as above:

$$y_{n+1} - P_{n+1} = y_n - P_n + h \left\{ \begin{array}{l} \frac{P\partial f(x_n, P_n)}{\partial P_n}(x_n - P_n) \\ +\frac{Q\partial f'(x_n, P_n)}{\partial P_n}(x_n - P_n) \\ +\frac{R\partial f'(x_n, P_n)}{\partial P_n}(x_n - P_n) \\ +\frac{S\partial f'(x_n, P_n)}{\partial P_n}(x_n - P_n) \end{array} \right\} \quad 4.6$$

$$y_{n+1} - P_{n+1} = y_n - P_n$$

$$+h \left\{ \begin{array}{l} \sup\limits_{(x_n, P_n) \in D} \frac{P\partial f(x_n, P_n)}{\partial P_n}(x_n - P_n) \\ + \sup\limits_{(x_n, P_n) \in D} \frac{Q\partial f'(x_n, P_n)}{\partial P_n}(x_n - P_n) \\ + \sup\limits_{(x_n, P_n) \in D} \frac{R\partial f'(x_n, P_n)}{\partial P_n}(x_n - P_n) \\ + \sup\limits_{(x_n, P_n) \in D} \frac{S\partial f'(x_n, P_n)}{\partial P_n}(x_n - P_n) \end{array} \right\} \quad 4.7$$

$$y_{n+1} - p_{n+1} = y_n - p_n + h \left\{ \begin{array}{c} PL(x_n, p_n) \\ +QL_1(x_n, p_n) \\ +PL_2(x_n, p_n) \\ +SL_3(x_n, p_n) \end{array} \right\} \quad 4.8$$

Taking the absolute value of both sides of (4.8) gives

$$|y_{n+1} - p_{n+1}| \le |y_n - p_n| + h|PL + Q\,L_1 + R\,L_2 + S\,L_3||x_n - p_n| \quad 4.9$$

let $N = |PL + Q\,L_1 + R\,L_2 + S\,L_3|$

and $y(x_0) = \zeta$ and $p(x_0) = \zeta^*$, given $\varepsilon > 0$. Then

$$|y_{n+1} - p_{n+1}| \le N|y_n - p_n| \quad 4.10$$

And

$$|y_{n+1} - p_{n+1}| \leq N|\zeta - \zeta^*| \text{ for every } \varepsilon > 0 \qquad 4.11$$

Then we conclude that our scheme is stable and hence convergent.

5.0 CONCLUSION AND RECOMMENDATION; we have presented comprehensive detail of the convergence theorem for a newly developed non-linear method for numerical integration of ordinary differential equations. It is proved that the scheme is convergent.

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