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Subordination Results for Sakaguchi Type Functions

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ABSTRACT *In this article, we introduce a new subclass L_s (α,β,t) of analytic function using Sakaguchi type functions. We investigate characterization and subordination results and we discuss several interesting consequences of these results.*

KEYWORDS : Sakaguchi type functions, Convolution, Subordination.

1. INTRODUCTION

Let A denote the class of all analytic univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

defined in the open unit disc $U = \{z: |z| < 1\}$.

Let K denote the class of functions that are convex in U and let $S(\alpha, t)$ be the subclass of A

consisting of functions given by (1.1) satisfying the condition

$$Re\left\{\frac{(1-t)z f'(z)}{f(z) - f(tz)}\right\} > \alpha, \quad (z\epsilon U, \quad 0 \le \alpha < 1, \quad |t| \le 1, \quad t \ne 1)$$

Also let $T(\alpha, t)$ denote the subclass of A consisting of all functions $f(z)$ such that $z f'(z) \in S(\alpha, t)$. These classes were introduced and studied by Owa et al [2]. The class $S(0, -1)$ was introduced by Sakaguchi in 1959 and the class $S(\alpha, -1)$ is called the class of Sakaguchi functions of order α.

Now we introduce a new subclass $L_s(\alpha, \beta, t)$ defined as follows.

Definition 1.1: A function $f(z)\epsilon$ A is said to be in the class $L_s(\alpha, \beta, t)$ if it satisfies

$$Re\left\{\frac{(1-t)zf'(z) + \beta(1-t)z^2 f''(z)}{(1-\beta)[f(z) - f(tz)] + \beta z[f(z) - f(tz)]'}\right\} > \alpha \qquad (1.2)$$

Where, $0 \le \alpha < 1$, $0 \le \beta < 1$, $|t| \le 1$, $t \ne 1$ and $z\epsilon U$.

REMARK: For different values of β and t this class $L_s(\alpha,\beta,t)$ reduces to the classes $S(\alpha,t)$, $T(\alpha,t)$, $S(\alpha,-1)$, $T(\alpha,-1)$, $S^*(\alpha)$, $K(\alpha)$ studied earlier by Owa et al[2], Sakaguchi[3].

2. CHARACTERIZATION RESULTS

In this section we prove the Characterization results for the functions in the class $L_s(\alpha,\beta,t)$.

THEOREM 2.1: If the function $f(z)\epsilon$ A , satisfies the inequality

$$\sum_{n=2}^{\infty}[1+(n-1)\beta][|n-u_n|+(1-\alpha)|u_n|]|a_n| \le 1-\alpha, \qquad (2.1)$$

Where $u_n=\sum_{k=0}^{n-1}t^k$, $0\le\alpha<1$, $0\le\beta<1$, $|t|\le 1$, $t\neq 1$ and $z\epsilon U$, then $f(z)\,\epsilon\, L_s(\alpha,\beta,t)$. The result is sharp.

PROOF: To prove the result it suffices to show that

$$\left|\frac{(1-t)zf'(z)+\beta(1-t)z^2f''(z)}{(1-\beta)[f(z)-f(tz)]+\beta z[f(z)-f(tz)]'}-1\right|<1-\alpha.$$

Since

$$\left|\frac{(1-t)zf'(z)+\beta(1-t)z^2f''(z)}{(1-\beta)[f(z)-f(tz)]+\beta z[f(z)-f(tz)]'}-1\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta](n-u_n)\,a_n\,z^n}{z+\sum_{n=2}^{\infty}[1+(n-1)\beta]u_n\,a_n\,z^n}\right|<\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n|\,|a_n|\,|z^{n-1}|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n|\,|a_n|\,|z^{n-1}|}$$

$$<\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n|\,|a_n|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n|\,|a_n|}.$$

We see that

$$\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n|\,|a_n|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n|\,|a_n|}<1-\alpha$$

$$\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n|\,|a_n| < (1-\alpha)\left(1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n|\,|a_n|\right)$$

$$\sum_{n=2}^{\infty}[1+(n-1)\beta][|n-u_n|+(1-\alpha)|u_n|]|a_n|\le 1-\alpha \quad .$$

REMARK: Suitable choices of β yield the coefficient inequalities for the functions belonging to the classes $S(\alpha,t)$ and $T(\alpha,t)$ derived in [2].

3. SUBORDINATION RESULTS

In this section we prove the Subordination results. To prove our result we require the following Definitions and also a related result due to wilf [4].

Definition 3.1: Given two functions f and g in the class A, where f is given by (1.1) and g is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or Convolution) $f * g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \qquad (z \epsilon U).$$

Definition 3.2: For two functions f and g analytic in U, we say that the function f is subordinate to g in U and write $f \prec g$, if there exists a Schwarz function ω, which is analytic in U, with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$, $z \epsilon U$.

Definition 3.3: A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z), \qquad (z \epsilon U, \; a_1 = 1).$$

Lemma 3.4:[4] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$Re\{1 + 2\sum_{n=1}^{\infty} b_n z^n\} > 0$, $z \epsilon U$.

THEOREM 3.5: Let $f(z) \epsilon A$ satisfy the inequality (2.1), and suppose that $g \epsilon K$, then

$$\frac{(1+\beta)[|1-t| + (1-\alpha)|1+t|]}{2[(1+\beta)[|1-t| + (1-\alpha)|1+t|] + (1-\alpha)} (f * g)(z) \prec g(z) \qquad (3.1)$$

Where $z \epsilon U$, $0 \le \alpha < 1$, $0 \le \beta < 1$, $|t| \le 1$, $t \ne 1$ and

$$Re\{f(z)\} > -\frac{(1+\beta)[|1-t| + (1-\alpha)|1+t|] + (1-\alpha)}{(1+\beta)[|1-t| + (1-\alpha)|1+t|]}, \; z \epsilon U \qquad (3.2).$$

The constant factor

$$\frac{(1+\beta)[|1-t| + (1-\alpha)|1+t|]}{2[(1+\beta)[|1-t| + (1-\alpha)|1+t|] + (1-\alpha)]}$$

in the subordination result (3.1) cannot be replaced by a longer one.

PROOF: Let f defined by (1.1) be in the class $L_s(\alpha, \beta, t)$ and suppose that

$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \epsilon K$. Then we have

$$\frac{(1+\beta)[|1-t| + (1-\alpha)|1+t|]}{2[(1+\beta)[|1-t| + (1-\alpha)|1+t|] + (1-\alpha)]} (f * g)(z) \qquad (3.3)$$

$$= \frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]}(z+\sum_{n=2}^{\infty} a_n c_n z^n)$$

By definition (3.3) the subordination result (3.1) holds true if the sequence

$$\left\{\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} a_n\right\}_{n=1}^{\infty} \qquad (3.4)$$

is a subordinating factor sequence with $a_1 = 1$.

In view of Lemma (3.4) it is enough to prove the inequality:

$$Re\left\{1+2\sum_{n=1}^{\infty}\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} a_n z^n\right\} > 0, (z\, \epsilon U) \ (3.5).$$

Now,

$$Re\left\{1+\sum_{n=1}^{\infty}\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} a_n z^n\right\}$$

$$= Re\left\{1+\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} z + \sum_{n=2}^{\infty}\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} a_n z^n\right\}$$

$$\geq 1-\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} r - \sum_{n=2}^{\infty}\frac{[1+(n-1)\beta][|n-u_n|+(1-\alpha)|u_n|]}{[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} |a_n| r^n$$

$$> 1-\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} r - \frac{(1-\alpha)}{[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} r > 0$$

Then (3.5) holds in U. This proves the inequality (3.1). The inequality (3.2) follows from (3.1), upon setting

$$g(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \,\epsilon\, K$$

To prove the sharpness of the constant

$$\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]}$$

we consider the function f_0 defined by

$$f_0(z) = z - \frac{(1-\alpha)}{(1+\beta)[|1-t|+(1-\alpha)|1+t|]} z^2 \qquad (3.6)$$

From (3.1)

$$\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} f_0(z) \prec \frac{z}{1-z} \qquad (3.7)$$

For the function f_0 , it is easy to verify that

$$Min\left\{Re\left\{\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]} f_0(z)\right\}\right\} = -1/2$$

This shows that the constant

$$\frac{(1+\beta)[|1-t|+(1-\alpha)|1+t|]}{2[(1+\beta)[|1-t|+(1-\alpha)|1+t|]+(1-\alpha)]}$$

is the best possible, which completes the proof.

Remark: For β =0 and 1 the above result reduces to the results by Frasin and Maslina Darus [1].

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