Research Paper | Mathematics

Integral solutions of Ternary quadratic Diophantine equation x^2+xy+y^2=7z^2

R.Anbuselvi Associate Professor , Department of Mathematics, ADM College for women, Nagapattinam, Tamilnadu ,India

S.Jamuna Rani Asst Professor, Department of Computer Applications, Bharathiyar college of Engineering and Technology, Karaikal, Puducherry, India

ABSTRACT *The ternary quadratic Diophantine equation given by x^2+xy+y^2=7z^2 is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited*

KEYWORDS : Ternary quadratic, integral solutions, polygonal numbers.

Introduction

Ternary quadratic equations are rich in variety [1-3].For an extensive review of sizable literature and various problems, one may refer [1-16]. In this communication, we consider yet another interesting ternary quadratic equation $x^2+xy+y^2=7z^2$ and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special Polygonal numbers are presented.

Notations Used

- $t_{m,n}$- Polygonal number of rank 'n' with size 'm'
- P_n^m- Pyramidal number of rank 'n' with size 'm'

Methods of Analysis

The Quadratic Diophantine equation with three unknowns to be solved for its non zero distinct integral solutions is

$$x^2+xy+y^2=7z^2 \qquad (1)$$

On substituting the linear transformations

$$x=u+v\,; \quad y=u-v \qquad (2)$$

in (1), it leads to

$$3u^2+v^2=7z^2 \qquad (3)$$

We obtain different patterns of integral solutions to (1) through solving (3) which are illustrated as follows:

Pattern I

Equation (3) is satisfied by

$$\left.\begin{aligned} u&=4ab+b^2-3a^2\\ v&=2b^2-6a^2-6ab\\ z&=33a^2+b^2 \end{aligned}\right\} \qquad (4)$$

Substituting (4)and (3)in (2), the corresponding non-zero distinct integral solutions of (1) are given by

$$\left.\begin{aligned} x(a,b)&=x=3b^2-9a^2-2ab\\ y(a,b)&=y=3a^2-b^2+10ab\\ z(a,b)&=z=3a^2+b^2 \end{aligned}\right\} \qquad (5)$$

Properties

1. $x(a,a(a+1))+3y(a,a(a+1))-56P_a^5\equiv 0$
2. $x(a,(a+1)(a+2))+3y(a,(a+1)(a+2))-168P_a^3\equiv 0$
3. $x(a(a+1),(a+1)(a+2))+3y(a\,(a+1),(a+1)(a+2))-336F_{4,n-4}\equiv 0$
4. $x(4,b)+3z(4,b)-156\,t_{3,b}+t_{146,b}\equiv 0(mod\ \ 14)$
5. $x\,(a,a)+y(a,a)-z(a,a)\equiv 0$
6. $x\,(a,-a)+z(a,-a)\equiv 0$
7. $y\,(a,-a)-x(a,-a)+z(a,-a)\equiv 0$
8. $y(a,3)+z(a,3)-42\,t_{3,a}+t_{74,a}\equiv 0(mod\ \ 24)$
9. $3y(a^2+1,a)+x(a^2+1,a)=\ 28a(a^2+1)$ is represent a perfect number
10. $x(2,b)+3y(2,b)\equiv 0\ (mod\ \ 56)$
11. $x(2a,a)+y(2a,a)+156\,t_{3,a}-\ t_{146,a}\equiv 0(mod\ \ 6)$

Pattern II

Introducing the linear transformation

$$\left.\begin{aligned} u&=x+7T\\ v&=4v\\ z&=x+3T \end{aligned}\right\} \qquad (6)$$

The corresponding non zero distinct solutions of (1) are given by

$$\left.\begin{aligned} x(r,s) &= x = 63r^2 - s^2 + 14rs \\ y(r,s) &= y = 3s^2 - 21r^2 + 14rs \\ z(r,s) &= z = 21r^2 + s^2 + 6rs \end{aligned}\right\} \quad (7)$$

Properties

1. $y(1,s) + z(1,s) - 196\, t_{3,s} + t_{188,s} \equiv 0 \ (mod\ 16)$
2. $x(r,1) + z(r,1) - 40t_{3,r} - 128t_{3,r} \equiv 0 (mod\ 4)$
3. $x(r,1) - y(r,1) - 168t_{3,r} \equiv -4(mod\ \ 84)$
4. $x(a,a) + y(a,a) + z(a,a)$ is a perfect square.
5. $x(1{,}1) - y(1{,}1)$ is a sum of two squares
6. Each of the following expressions represents a nasty number

(i) $[y(1{,}1) + z(1{,}1)]$

(ii) $[x(a,a) + z(a,a) - 2y(a,a)]$

Pattern III

Equation (3) can be written as

$$3u^2 + v^2 = 7z^2 \quad (8)$$

Take

$$z = 3a^2 + b^2 \quad (9)$$

$$1 = \frac{(1 + i\sqrt{33})(1 - i\sqrt{33})}{4} \quad (10)$$

Using (9) and (10) in (8) and employing the method of factorization, define

$$v + i\sqrt{3}\, u = \frac{1}{2}(3i\sqrt{3}\ b^2 - b^2 \ - \ 9i\sqrt{3}\ a^2 + 3\, a^2 - 18ab - 2i\sqrt{3}\ ab)$$

Equating real and imaginary part, we get

$$\left.\begin{aligned} u &= \frac{1}{2}(f(a,b)) \\ &= \frac{1}{2}(3b^2 - 9a^2 - 2ab) \\ v &= \frac{1}{2}(g(a,b)) \\ &= \frac{1}{2}(3a^2 - b^2 - 18ab) \end{aligned}\right\} \quad (11)$$

$$z = 3a^2 + b^2$$

As our interest is on finding integer solutions, it is seen that the values of x,y and z are integers when both a and b are of the same parity. Thus by taking $a = 2A$, $b = 2B$ in (11) and substituting the corresponding values of u,v in (2) the non-zero integral solution of (1) are given by

$$\left.\begin{aligned} x(A,B) &= x = 4B^2 - 12A^2 - 40AB \\ y(A,B) &= y = 8B^2 - 24A^2 + 32AB \\ z(A,B) &= z = 12A^2 + 4B^2 \end{aligned}\right\} \quad (12)$$

Properties

1. $y(A,(A+1)) - 2x(A,(A+1)) - 224\, t_{3,A} \equiv 0$
2. $y(A, A(A+1)) - 2x(A, A(A+1)) - 224\, P_A^5 \equiv 0$
3. $y(A,(A+1)(A+2)) - 2x(A,(A+1)(A+2) - 672P_A^3 \equiv 0$
4. $y(A,(A+1)(A+2)(A+3)) - 2x(A,(A+1)(A+2)(A+3)) - 2688P_A^4 \equiv 0$
5. $x(A,A) - z(A,A)$ is expressed as a perfect square.
6. $y(1{,}1) - 2x(1{,}1)$ can be expressed as a difference of two perfect squares
7. $x\,(A,-A) + y(A,-A\,) + z(A,-A) \equiv 0$
8. $y\,(A,-A) + 3z(A,-A) \equiv 0$
9. $x(2,3) + y(2,3) + z(2,3) \equiv 0$
10. $2x(A^2,A) - y(A^2,A) + 112\ CP_A^6 \equiv 0$
11. Each of the following expression represents a nasty number

(i) $[2\, x(1{,}1)]$

(ii) $[\, z(2,3)]$

(iii) $[x(1,2) + y(1,\ 2) + 2(1{,}2)]$

(iv) $[x(1,2) -\ y(1,2) + z\,(1,2)]$

Pattern IV

$$x^2 + xy + y^2 = 7z^2$$

$$4x^2 + 4xy + 4y^2 = 28z^2$$

can be expressed as

$$(2x+y)^2 + 3y^2 = 28z^2$$

$$u^2 + 3v^2 = 28z^2 \quad (13)$$

$$\left.\begin{aligned} u &= 2x + y \\ v &= y \end{aligned}\right\} \quad (14)$$

$$z = a^2 + 3b^2 \qquad (15)$$

Using (15) and (14) in (13) and employing the method of factorization, define

$$(u + i\sqrt{3}\, v)(u - i\sqrt{3}\, v) = 28z^2$$

$$(u + i\sqrt{3}\, v)(u - i\sqrt{3}\, v) = (4 + 2i\sqrt{3}\,)(4 - 2i\sqrt{3}\,)(a + i\sqrt{3}b)^2(a - i\sqrt{3}b)^2$$

$$(u + i\sqrt{3}\, v) = (4a^2 - 12ab - 12b^2) + i\sqrt{3}(2a^2 + 8ab - b^2)$$

Equating real and imaginary part, we get

$$\left.\begin{aligned} u &= 4a^2 - 12b^2 - 12ab \\ v &= 2a^2 - b^2 + 8ab \end{aligned}\right\} \qquad (16)$$

Substituting (16) in (14) , we get

$$\left.\begin{aligned} x &= a^2 - 3b^2 - 10ab \\ y &= 2a^2 - 6b^2 + 8ab \\ z &= a^2 + 3b^2 \end{aligned}\right\} \qquad (17)$$

Properties

1. $y(A, (A + 1)) - 2x(A, (A + 1)) - 56t_{3,A} \equiv 0$
2. $y(A, A(A + 1)) - 2x(A, A(A + 1)) - 56P_A^5 \equiv 0$
3. $y(A, (A + 1)(A + 2)) - 2x(A, (A + 1)(A + 2)) - 168P_A^3 \equiv 0$
4. $y(A, (A + 1)(A + 2)(A + 3)) - 2x(A, (A + 1)(A + 2)(A + 3)) - 672P_A^4 \equiv 0$
5. $x(A, A) - z(A, A)$ is expressed as a perfect square.
6. $y(1,1) - 2x(1,1)$ can be expressed as a difference of two perfect squares
7. $3y(2, 2) - x(2, 2) \equiv 0$
8. $x(A, -A) + y(A, -A) + z(A, -A) \equiv 0$
9. $y(A, -A) - 2z(A, -A) \equiv 0$
10. $6y(B, B) - x(B, B) \equiv 0$ can be expressed as a perfect square.
11. Each of the following expression represents a nasty number
    (i) $[z(3,3) + y(3,3) - x(3,3)]$
    (ii) $[z(2,3) - y(2,3) + x(2,3)]$

Conclusion

To conclude, one may search for other patterns of solutions and their corresponding properties.

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