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Analysis of Some Results on Fuzzy Connected Space And Local Connectedness.

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ABSTRACT : *In this paper, we analyze some results in fuzzy connected spaces. Among the results obtained we can mention the good extension of local connectedness.*

KEYWORDS : : Fuzzy connected, Strong Connectedness, fuzzy locally connectedness, Supper connectedness, quasi-coincident.

Introduction:1.1 The fundamental concept of a fuzzy sets was introduced by Zadeh[1].Subsequently, Chang [2] defined the notion of fuzzy topology. Since the various aspects of topology were investigated and carried out in fuzzy sense by several authors of this field. The local properties of space which may also be in certain cases the properties of the whole spaces are important field for study in both general and fuzzy topology.In general topology, by introducing the notion of ideal, Kuratowski[3], Vaidyanathaswamy[4,5] and several other authors carried out such analysis. The notion of fuzzy points and Q- neighborhood of a fuzzy point introduces the scope of such analysis in fuzzy topology[6, 7]. In 1997 Sarkar [8] extended the concept of ideals in fuzzy settings and defined the local function in fuzzy topology. With the emergence of the fundamental paper [1] by Zadeh in 1965 number of apers have appeared in literature featuring the application of fuzzy sets to pattern recognition, decision problems, function approximation, system theory, fuzzy logic, fuzzy topology, etc.[9], [10]. In this note, our interests are in the study of certain concepts in fuzzy topology. In this paper we introduce new results in fuzzy connected spaces. Among the results obtained we can mention the good extension of local connectedness.

Preliminaries:2.1

Definition 2.1.1 A Fuzzy topology is a family τ of fuzzy sets in X which satisfies the following conditions:

(i) $\emptyset, X \in \tau$.

(ii) If $A, B \in \tau$, then $A \cap B \in \tau$.

(iii) If $A_i \in \tau$, for each $i \in I$, then $\bigcup_i A_i \in \tau$.

τ is called a fuzzy topology for X, and the pair $(X, \tau)$is called a fuzzy topological space. If the null fuzzy set 0 and the whole fuzzy set 1 belongs to τ and τ is closed with respect to any union and finite intersection. The members of τ are called fuzzy open sets and their complements are called fuzzy closed sets. The Closure [11] of a fuzzy set A of X denoted by $Cl(A)$, is the intersection of all fuzzy closed sets which contains A. The interior [11] of a fuzzy set A of X denoted by $Int(A)$ is the union of all fuzzy sets of X contained in A.

Definition 2.1.2 [11] Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(Y)$. Then the inverse image of B written as $f^{-1}(B)$ is a fuzzy set in X whose membership function defined by $\mu_{f^{-1}(B)}(X) = \mu_B\{f(x)\}$ for all x in X.

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Definition 2.1.3 [11] A function $'f'$ from a fuzzy topological space (X,τ) to a fuzzy topological space (Y,σ) is $F-continuous$ if and only if the inverse of each $\sigma-open$ fuzzy set is $\tau-open$.
Definition 2.1.4 A fuzzy topological space X is said to be disconnected if $X = A \cup B$, where A and B are non-empty open fuzzy sets in X such that $A \cap B = \emptyset$.
Hence, a connected space is defined as follows:
Definition 2.1.5: A fuzzy topological space X is said to be connected if X cannot be represented as the union of two non-empty, disjoint open fuzzy sets on X.
Definition 2.1.6: A fuzzy set A in fuzzy topological space (X,τ) is said to be quasi-coincident [11] with a fuzzy set B denoted by AqB if there exists a point $x \in X$ such that
$A(x)+B(x)>1$. The negation of the statement is written as $\sim(AqB)$.
Definition 2.1.7 A fuzzy set V in a fuzzy topological space (X,τ) is called a Q-neighborhood [11] of a fuzzy point X_β if there exists a fuzzy open set U of X such that $X_\beta qU \le V$.
Definition 2.1.8 [12] A fuzzy topological space X is said to be fuzzy connected if it has no proper fuzzy closed-open set.
Definition 2.1.9 [12] A fuzzy topological space X is said to be fuzzy super-connected if X does not have non-zero fuzzy open sets $\lambda+\mu \le 1$.
Definition 2.1.10 [12] A fuzzy topological space X is said to be fuzzy strong connected if it has no non- zero fuzzy closed sets f and k such that $f+k \le 1$.
Definition 2.1.11 [13] A fuzzy topological space X is said to be c-zero dimensional if for every crisp fuzzy point x_1 in X and every fuzzy open set μ containing x_1 there exists a crisp clopen fuzzy set δ in X such that $x_1 \le \delta \le \mu$.
Definition 2.1.12 [13] A fuzzy topological space X is said to be totally c_i-disconnected ($i = 1,2,3,4$) if the support of every non zero c_i-connected fuzzy set ([1, Definition 2.7]) in X is a singleton.
Definition 2.1.13 [13] A fuzzy topological space X is said to be strong c-zero dimensional if it is not c-connected between any pair of its disjoint fuzzy closed sets.
Definition 2.1.14 [14] A fuzzy topological space X is said to be fuzzy locally connected at a fuzzy point x_α in X if for every fuzzy open set μ in X containing x_α, there exists a connected fuzzy open set δ in X such that $x_1 \le \delta \le \mu$.
Definition 2.1.15 [14] Let (X,τ) be a fuzzy topological space, suppose $A \subset X$ and let $\tau_A = \{\mu|A : \mu \in \tau\}$. Then (A,τ_A) is called a fuzzy subspace of (X,τ). In short we shall denote (A,τ_A) by A. A fuzzy subspace A is said to be a fuzzy open subspace if its characteristic function χ_A is fuzzy open in (X,τ).
Let μ be a fuzzy set in X and let $A \subset X$. Then we denote $\mu|A$ by μ^a. In particular, if μ is a fuzzy point x_α in X, then we denote the fuzzy set x_{α_A} by x_α^a.
Definition 2.1.16 [13] A fuzzy topological space (X,τ) is said to be c-connected between fuzzy sets μ and δ if there exists no crisp closer-open fuzzy η such that $\mu \le \eta \le 1_X - \delta$.
Definition 2.1.17 [12] Let (X,τ) be a fuzzy topological space, let $A \subset X$. Then A is said to be a fuzzy connected subset of X if A is a fuzzy connected space as a fuzzy subspace of X.
Definition 2.1.18 A fuzzy topological space (X,τ) is said to be locally fuzzy super connected (locally fuzzy strong connected) at a fuzzy point x_a in X if for every fuzzy open set μ in X containing x_a there exists a fuzzy super connected (fuzzy strong connected) open set η in X such that $x_a \le \eta \le \mu$.

3.1 Here we analyze some results in fuzzy connected spaces. Among the results obtained we can mention the good extension of local connectedness.

Theorem 3.1.1: The F −continuous image of a connected fuzzy space X to a fuzzy topological space is connected.

Proof: Let $f: X \to Y$ be a F −continuous map of X to Y where (X,τ) and $(Y,\tau_{f(X)})$ are fuzzy topological spaces, X is connected.

Suppose $f(X)$ is disconnected. Then there exist non-empty open fuzzy open sets G and $H \in \tau_{f(X)}$ such that $f(X) = G \cup H$. This implies that G and H are obtained from non-empty open fuzzy sets say G_y and H_y in Y such that $G = G_y \cap f(X)$; $H = H_y \cap f(X)$.

We shall show that $f^{-1}(G)$ and $f^{-1}(H)$ give a disconnection for X. That is, $X = f^{-1}(G) \cup f^{-1}(H)$.

For by Definition 2.1.4,

$$\min\left(\mu_{f^{-1}(G)}(x), \mu_{f^{-1}(H)}(x)\right) = \max\left(\mu_G\big(f(x)\big), \mu_H\big(f(x)\big)\right)$$
$$= 1\left(since\ G \cup H = f(X)\right),$$
$$= \mu_x(x).$$

This is a contradiction to X being connected.

Hence, $f(X)$ is connected.

Theorem 3.1.2 A topological space (X,τ) is locally connected if and only if $(X,\omega(\tau))$ is locally connected.(where $\omega(\tau)$ is the set of all lower semi-continuous functions from (X,τ) to the unit interval $I = [0,1]$).

Proof: Let μ be a fuzzy open set in $\omega(\tau)$ containing a fuzzy point x_α. Since μ is a lower semi continuous function, by local connectedness of (X,τ) there exists an open connected set U in X containing x and contained in the support of μ. Now χ_U is the characteristic function of U and it is lower semicontinuous, hence $\chi_U \wedge \mu$ is a fuzzy open set in $\omega(\tau)$. We claim $\delta = \chi_U \wedge \mu$ is a fuzzy connected set containing x_α. If not then by [15, Theorem 3.1], there exist a non zero lower semicontinuous function μ_1, μ_2 in $\omega(\tau)$ such that $\mu_1|\delta + \mu_2|\delta = 1$.

Now Supp $\delta = U$ and Supp μ_1, Supp μ_2 are open sets in τ such that $U \subset$ Supp $\mu_1 \cup$ Supp μ_2,

Hence, $U \cap$ Supp $\mu_1 \neq \emptyset$ and $U \cap$ Supp $\mu_2 \neq \emptyset$ and consequently

($U \cap$ Supp μ_1)∪($U \cap$ Supp μ_2)=$U \cap$ (Supp $\mu_1 \cup$ Supp μ_2) $= U$ is not connected.

Conversely, let U be an open set in τ containing $x, x_\alpha \in \chi_{U,}$ (χ_U is a characteristic function of U),

$\chi_U$is a fuzzy open set in $\omega(\tau)$. By fuzzy connectedness of $(X,\omega(\tau))$ there exists a fuzzy open connected set μ in $\omega(\tau)$ such that $x_\alpha \leq \mu \leq \chi_{U.}$

We claim that Supp μ is connected ($x \in$Supp $\mu \subset U$). If not there exist two non empty open sets $G_1, G_2 \in \tau$ such that

Supp $\mu = G_1 \bigcup G_2$ and $G_1 \bigcap G_2 = \emptyset$.

It is clear that $\chi_{G_1} + \chi_{G_2} = 1_\mu$,

Which is a contradiction, because μ is fuzzy connected.

Theorem 3.1.3 If G is a subset of a fuzzy topological space (X,τ) such that μ_G(Characteristic function of a subset G of X) is fuzzy open in X, then if X is a super connected space then $G$is a fuzzy super connected space.

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Proof: Suppose that G is not a super connected space. Then by [12, Theorem 6.1] there exist fuzzy open sets λ_1, λ_2 $in\ X$ such that $\lambda_1 \big| G \neq 0$, $\lambda_2 \big| G \neq 0$
and $\lambda_1 \big| G + \lambda_2 \big| G \leq 1$,
and therefore $\lambda_1 \wedge \mu_G + \lambda_1 \wedge \mu_G \leq 1$.
Then X is not a fuzzy super connected space, a contradiction.

Theorem 3.1.4 If A and B are fuzzy strong connected subsets of a fuzzy topological space (X, τ) and $\overline{\mu_B} \big| A \neq 0$ or $\overline{\mu_A} \big| B \neq 0$ then $A \vee B$ is a fuzzy strong connected subset of X where μ_A, μ_B are the characteristic of A and B respectively.

Proof: Suppose $Y = A \vee B$ is not a fuzzy strong connected subset of X. Then there exist fuzzy closed sets δ and λ such that $\delta \big| Y \neq 0$ and $\lambda \big| Y \neq 0$ and $\delta \big| Y + \lambda \big| Y \leq 1$. Since A is a fuzzy strong connected subset of X, then either $\delta \big| A = 0$ or $\lambda \big| A = 0$. Without loss of generality assume that $\delta \big| A = 0$. In this case, since B is also fuzzy strong connected, we have

$\delta \big| A = 0, \quad \lambda \big| A \neq 0, \quad \delta \big| B \neq 0, \quad \lambda \big| B = 0$

and therefore $\lambda \big| A + \overline{\mu_B} \big| A \leq 1$ .... (1)

If $\overline{\mu_B} \big| A \neq 0$, then $\lambda \big| A \neq 0$ with (1) imply that A is not a fuzzy connected subset of X. In the same way $\overline{\mu_A} \big| B \neq 0$ then $\delta \big| B \neq 0$ and $\lambda \big| B + \overline{\mu_A} \big| B \leq 1$ imply that B is not a fuzzy strong connected subset of X, a contradiction.

Theorem 3.1.5 If A and B are subsets of a fuzzy topological space (X, τ) and $\mu_A \leq \mu_B \leq \overline{\mu_A}$, then A is a fuzzy strong connected subset of X then B is also fuzzy strong connected.

Proof: Let B be not fuzzy strong connected. Then there exist two non zero fuzzy closed sets $f|B$ and $k|B$ such that $f|B + k|B \leq 1$. ....(2)
If $f|A = 0$ then $f + \mu_A \leq f + \mu_B \leq f + \overline{\mu_A}$. ....(3)
Hence $f + \mu_B \leq 1$, thus $f|B = 0$, a contradiction, and therefore $f|A \neq 0$.
Similarly we can show that $k|A \neq 0$. By (2) and with the relation $\mu_A \leq \mu_B$ we conclude that $f|A + k|A \leq 1$,
So A is not fuzzy strong connected, which is again a contradiction.

Theorem 3.1.6 A fuzzy topological space (X, τ) is locally fuzzy connected if and only if every fuzzy open subspace of X is locally connected.

Proof: Let A be a fuzzy open subspace of X and let η be a fuzzy open set in X. To prove A is fuzzy connected, let x_α^a be a fuzzy point in A and let $\eta|A$ be a fuzzy open set in A containing x_α^a. We must prove that there exists a connected fuzzy open set $\mu|A$ in A such that $x_\alpha^a \leq \mu|A \leq \eta|A$.
Clearly, the fuzzy point x_α in X lies in η. Since X is locally fuzzy connected, there exists an open fuzzy connected μ such that $x_\alpha \leq \mu \leq \eta$ and $\mu \leq \eta \wedge \chi_{A.}$
It is easy to prove that $x_\alpha^a \leq \mu|A \leq \eta|A$.
If $\mu|A$ is not fuzzy connected, then there exists a proper fuzzy close-open $\lambda|A$ in $\mu|A$ ($\lambda$is proper fuzzy close-open in μ). This is a contradiction with the fact that μ is fuzzy connected and hence $A$is fuzzy connected.

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