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Weighted Weibull Length Biased Distribution and A Change Point Model: A Special Case of Bayesian Approach and Lengthbiasedness

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ABSTRACT
Here, we have developed a change point model related to Weibull Length Biased Life Time Model. We have made an attempt to study a special case of length biasedness of Weibull Life Time Model where the Weibull Length Biased Model gets transformed into Weighted Weibull Length Biased Distribution. Further, we have obtained the posterior densities of θ_1,θ_2,β and 'm' for this model and then we have obtained the Bayes estimates of θ_(1,) θ_2,β and 'm' under asymmetric loss functions. We have also studied the sensitivity of Bayes estimates with respect to change in the prior of the parameters for the model. Finally, we have given the conclusions on the basis of our study.

KEYWORDS : Weighted Weibull Length Biased Distribution, Length Biasedness, Life Time Model, Bayes Estimates, Change Point, Loss Functions.

We know that the length-biased distribution has very widely applications in biomedical area such as family history and disease, survival and intermediate events and latency period of AIDS due to blood transfusion. Gupta and Akman were first to study about this in 1995. They developed and presented an article on the study of human families and wildlife populations. It included a list of the most common forms of the weight function useful in scientific and statistical literature which was deeply studied by Patill and Rao in 1978 and developed further in 1986. Moreover, there were some basic theorems for weighted distributions and size-biased as a special case. Finally, the conclusion was made that the length-biased version of some mixture of discrete distributions arises as a mixture of the length-biased version of these distributions. A significant work was done to characterize relationships between original distributions and their length biased versions and therefore it became necessary to work further on this aspect. It was in the year 1978 that Patill and Rao gave a table for some basic distributions and their length biased forms such as Beta, Gamma, Lognormal and Pareto distributions. The weighted version of the bivariate three-parameter logarithmic series distribution was studied by Gupta and Tripathi in 1996. It has wide applications in many fields such as ecology, social and behavioral sciences and species abundance studies.

Let us assume that X be a random variable following the Weibull distribution with pdf as under:

$g(x) = \theta \beta x^{\beta-1} \exp(-\theta x^{\beta})$

where $x \geq 0, \beta > 0, \theta > 0$ (1)

Here β is the shape parameter and θ scale parameter. We know that Weibull distribution is very flexible and this is due to its application in modeling in both the cases, viz. increasing (β > 1) as well as decreasing (β < 1) failure rates.

Moreover, we have $E(X) = \Gamma\beta^{-1}/\beta\theta^{\beta^{-1}}$.

Let T be a non negative random variable, T is said to have the Weibull length-biased distribution it will be abbreviated as WLB if its density function is given by:

$$f(t) = \frac{\beta^2 \theta^{(\frac{1}{\beta}+1)} t^{\beta} e^{-\theta t^{\beta}}}{\Gamma\frac{1}{\beta}} \text{ where } \beta, \theta > 0 \text{ and } t > 0 \quad (2)$$

The density (2) can be obtained by combining the definition of the length- biased distribution given by:

$$f(t) = \frac{tg(t)}{E(t)} \quad (3)$$

It can be explained as follows:

Suppose that the lifetime of a given sample of items follows Weibull Distribution and the density of the original distribution given in (2). As per the property of this distribution, the item doesn't have the same chance of being selected but each one is selected according to its length or life length then the resulting distribution is neither Weibull nor Weibull Length Biased. The resultant distribution becomes the special case which we term as Weighted Weibull Length-Biased Distribution.

The reliability function is given by:

$$R(t) = 1 - \frac{\beta\gamma\left(\frac{1}{\beta}+1,\theta t^{\beta}\right)}{\Gamma\frac{1}{\beta}} \tag{4}$$

Here, we note that the numerator represents the incomplete gamma function as:

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt \tag{5}$$

2. PROPOSED CHANGE POINT MODEL:

Let $T_1, T, T_3 \ . \ . \ . \ , T_n$ (n ≥ 3) be a sequence of observed life time data. Let first 'm' obsesrvations $T_1, T_2, \ . \ . \ . \ . \ , T_m$ have come from the Weighted Weibull Length Biased distribution with probability density function WWLB (β,θ_1).

$$f(t_i) = \beta\, \theta_1{}^2\ t_i{}^{2\beta-1}\ e^{-\theta_1 t_i{}^{\beta}} \text{ where } i=1,2,\ldots,m$$

and later (n-m) observations coming from the Weighted Weibull Length Biased with probability density function WWLB (β,θ_2)

$$f(t_i) = \beta\, \theta_2{}^2\ t_i{}^{2\beta-1}\ e^{-\theta_2 t_i{}^{\beta}} \text{ where } i=m+1,\ldots,n \tag{6}$$

where β , θ_1, θ_2>0

The likelihood function, given the sample information

$\underline{T} = (T_1, T_2, \ldots T_m, T_{m+1}, \ldots T_n)$ is

$$L\left(\theta_{1,}\theta_{2,}\beta, m\middle|\ \underline{T}\right) = \beta^n\theta_1{}^{2m}A_1{}^{2\beta-1}e^{-\theta_1 A_2}\theta_2{}^{2(n-m)}e^{-\theta_2 A_3} \tag{7}$$

where $A_1 = \prod_{i=1}^{n} t_i$

A_2=$A_2(m,\beta\ /t_i) = \sum_{i=1}^{m} t_i{}^{\beta}$

$$A_3\text{=}A_3(m,\beta\ /t_i) = \sum_{i=m+1}^{n} t_i{}^{\beta} \tag{8}$$

3. POSTERIOR DISTRIBUTION FUNCTIONS USING INFORMATIVE PRIOR:

Let us assume that the marginal prior distribution of 'm' be following discrete uniform distribution over the set {1, 2,…,n – 1} citing the research work of Broemeling et al.(1987).

So we take $g\ (\mathrm{m}) = \frac{1}{n-1}$ (9)

We cite the research work of Calabria and Pulcini carried out in 1992 and suppose the marginal prior distribution on β to be uniform over the interval β_1,β_2 as under:

$$g\ (\beta) = \frac{1}{\beta_2-\beta_1} \quad \beta_1 \leq \beta \leq \beta_2 \tag{10}$$

Further we also cite the phenomenal research work of N. Sanjari Farsipour and H. Zakerzadeh done in the year 2005. As per their work, we assume that the scale parameters θ_1 and θ_2 are unknown and we take the Inverted Gamma prior with probability density functions with respective means values μ_1, μ_2 and common standard deviation σ as under:

$$g\left({}^{\theta_1}/_{\beta}\right)=\frac{a_1{}^{\beta}}{\Gamma\beta}\ \theta_1{}^{-(\beta+1)}\ e^{-a_1/\theta_1}$$

$$g\left({}^{\theta_2}/_{\beta}\right)=\frac{a_2{}^{\beta}}{\Gamma\beta}\ \theta_2{}^{-(\beta+1)}\ e^{-a_2/\theta_2} \quad a_i, \beta > 0, \theta_i > 0\ , \mathrm{i} = 1,2 \tag{11}$$

It is quite clear that this prior distribution has significant advantages over many other distributions because of its analytical tractability, richness and interpretability.

Let the prior information be given in terms of the prior means μ_1, μ_2 and variances$\sigma_1{}^2$,$\sigma_2{}^2$. Then $\mu_i = E[\theta_i]=\frac{a_i}{\beta-1}$ and $\sigma_i{}^2 = \frac{a_i{}^2}{(\beta-1)^2(\beta-2)}$ i=1, 2, which gives

$$a_i = \mu_i(\frac{\mu_i^2}{\sigma_i^2}+1) \text{ and } \beta=2+(\frac{\mu_i^2}{\sigma_i^2}) \text{ where i=1, 2} \quad (12)$$

Thus if we have prior knowledge of μ_1, μ_2 and σ_1^2, σ_2^2 then the Inverted gamma parametersa_i, β, where i=1, 2 can be obtained from (12).

We assume that θ_1, θ_2, β and m are priori independent. The joint prior density will be:

$$g(\theta_1, \theta_2, \beta, m) = \frac{1}{n-1}\frac{1}{\beta_2-\beta_1}\frac{a_1^\beta}{\Gamma\beta}\,\theta_1^{-(\beta+1)}\,e^{-a_1/\theta_1}\frac{a_2^\beta}{\Gamma\beta}\,\theta_2^{-(\beta+1)}\,e^{-a_2/\theta_2}$$

$$=K_1\frac{a_1^\beta}{\Gamma\beta}\,\theta_1^{-(\beta+1)}\,e^{-a_1/\theta_1}\frac{a_2^\beta}{\Gamma\beta}\,\theta_2^{-(\beta+1)}\,e^{-a_2/\theta_2}$$

$$\text{where } K_1 = \frac{1}{n-1}\frac{1}{\beta_2-\beta_1} \quad (13)$$

The joint posterior density of parameters $\theta_1, \theta_2, \beta$, and m is obtained using the likelihood function (7) and the joint prior density of the parameters in (13) as under:

$$g(\theta_1, \theta_2, \beta, m \mid \underline{\mathrm{T}}) = \frac{L(\theta_1, \theta_2, \beta, m|\underline{\mathrm{t}})g(\theta_1, \theta_2, \beta, m)}{h(\underline{\mathrm{t}})}$$

$$= \beta^n\theta_1^{2m}A_1^{2\beta-1}e^{-\theta_1A_2}\theta_2^{2(n-m)}e^{-\theta_2A_3}\,K_1\frac{a_1^\beta}{\Gamma\beta}\,\theta_1^{-(\beta+1)}\,e^{-a_1/\theta_1}\frac{a_2^\beta}{\Gamma\beta}\,\theta_2^{-(\beta+1)}\,e^{-a_2/\theta_2}\,h^{-1}(\underline{\mathrm{T}})$$

$$=K_1\frac{a_1^\beta}{\Gamma\beta}\frac{a_2^\beta}{\Gamma\beta}A_1^{2\beta-1}\beta^n\,\theta_1^{2m-\beta-1}e^{-(\theta_1A_2+a_1/\theta_1)}\theta_2^{2(n-m)-\beta-1}e^{-(\theta_2A_3+a_2/\theta_2)}\,h^{-1}(\underline{\mathrm{T}}) \quad (14)$$

Here, $h(\underline{\mathrm{T}})$ is the marginal posterior density of $\underline{\mathrm{T}}$.

$$h(\underline{\mathrm{T}}) = \sum_{m=1}^{n-1}\int_{\beta_1}^{\beta_2}\int_0^\infty\int_0^\infty L(\theta_1, \theta_2, \beta, m|\underline{\mathrm{t}})g(\theta_1, \theta_2, \beta, m)\,d\theta_1\,d\theta_2\,d\beta$$

$$= \sum_{m=1}^{n-1}K_1\int_{\beta_1}^{\beta_2}\frac{a_1^\beta}{\Gamma\beta}\frac{a_2^\beta}{\Gamma\beta}\left(\frac{1}{\Gamma\frac{1}{\beta}}\right)^n A_1^{2\beta-1}\beta^n\{\int_0^\infty\theta_1^{2m-\beta-1}e^{-(\theta_1A_2+a_1/\theta_1)}\,d\theta_1$$

$$\int_0^\infty {\theta_2}^{2(n-m)-\beta-1} e^{-(\theta_2 A_3+ {a_2}/{\theta_2})}\, d\theta_2 \}\, d\beta$$

$$=\sum_{m=1}^{n-1} K_1 \int_{\beta_1}^{\beta_2} \frac{{a_1}^{\beta}}{\Gamma\beta}\frac{{a_2}^{\beta}}{\Gamma\beta} {A_1}^{2\beta-1} \beta^n I_1(m,\beta)\, I_2(m,\beta)\, d\beta \qquad (15)$$

where $I_1(m,\beta)= \int_0^\infty {\theta_1}^{2m-\beta-1} e^{-(\theta_1 A_2+ {a_1}/{\theta_1})}\, d\theta_1$

$$= 2{A_2}^{-[2m-\beta]/2} \left[\frac{1}{a_1}\right]^{-[2m-\beta]/2} \text{Bessel K}\, [-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}] \quad (16)$$

$$I_2(m,\beta)= \int_0^\infty {\theta_2}^{2(n-m)-\beta-1} e^{-(\theta_2 A_3+ {a_2}/{\theta_2})}\, d\theta_2$$

$$= 2{A_3}^{-[2(n-m)-\beta]/2} \left[\frac{1}{a_2}\right]^{-[2(n-m)-\beta]/2} \text{Bessel K}\, [-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}] \quad (17)$$

where Bessel K$[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}]$ and

Bessel K$[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]$ are defined as

$$\int_0^\infty b^{-1-m}\; e^{-(bc+d/b)} db = 2(c/d)^{m/2}\, Bessel\, K[m, 2\sqrt{c}\sqrt{d}] \qquad (18)$$

The marginal density of θ_1 say $g(\theta_1|\underline{T})$is as,

$$g(\theta_1|\underline{T}) = K_1 \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \int_0^\infty g(\theta_1,\theta_2,\beta|\underline{T})\, d\theta_2\, d\beta$$

$$= K_1 \sum_{m=1}^{n-1} {\theta_1}^{2m-\beta-1} e^{-(\theta_1 A_2+ {a_1}/{\theta_1})}\, 2{A_3}^{-[2(n-m)-\beta]/2}$$

$$\left[\frac{1}{a_2}\right]^{-[2(n-m)-\beta]/2} \text{Bessel K}[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}] \int_{\beta_1}^{\beta_2} \frac{{a_1}^{\beta}}{\Gamma\beta}\frac{{a_2}^{\beta}}{\Gamma\beta} {A_1}^{2\beta-1} \beta^n\, d\beta\, h^{-1}(\underline{T}) \quad (19)$$

where Bessel K$[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]$ is same as in (18).

The marginal density of θ_2 say $g(\theta_2|\underline{\mathrm{T}})$ is as,

$$g(\theta_2|\underline{\mathrm{T}}) = K_1 \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \int_0^{\infty} g(\theta_1, \theta_2, \beta|\underline{\mathrm{T}})\, d\theta_1\, d\beta$$

$$= K_1 \sum_{m=1}^{n-1} \theta_2^{\,2(n-m)-\beta-1} e^{-(\theta_2 A_3 + a_2/\theta_2)}\, 2A_2^{-[2m-\beta]/2}$$

$$\left[\frac{1}{a_1}\right]^{-[2m-\beta]/2} \text{Bessel K}\,[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}]$$

$$\int_{\beta_1}^{\beta_2} \frac{a_1^{\beta}}{\Gamma\beta} \frac{a_2^{\beta}}{\Gamma\beta} A_1^{2\beta-1} \beta^n\, d\beta\, h^{-1}(\underline{\mathrm{T}}) \qquad (20)$$

where Bessel K$[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}]$ is same as in (18).

The marginal density of β, say $g(\beta|\underline{\mathrm{T}})$ will be as under:

$$g(\beta|\underline{\mathrm{T}}) = K_1 \sum_{m=1}^{n-1} \int_0^{\infty} \int_0^{\infty} g(\theta_1, \theta_2, \beta|\underline{\mathrm{T}})\, d\theta_1\, d\theta_2$$

$$= K_1 \sum_{m=1}^{n-1} 2A_2^{-[2m-\beta]/2} \left[\frac{1}{a_1}\right]^{-[2m-\beta]/2} \text{Bessel K}\,[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}]\, 2A_3^{-[2(n-m)-\beta]/2}$$

$$\left[\frac{1}{a_2}\right]^{-[2(n-m)-\beta]/2} \text{Bessel K}[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]\, \frac{a_1^{\beta}}{\Gamma\beta} \frac{a_2^{\beta}}{\Gamma\beta} A_1^{2\beta-1} \beta^n\, h^{-1}(\underline{\mathrm{T}}) \qquad (21)$$

where Bessel K$[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}]$ and

Bessel K $[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]$ are same as in (18).

Marginal posterior density of m say, $g(\boldsymbol{m}\,|\underline{\mathrm{T}})$ is taken as under:

$g(\boldsymbol{m}\,|\underline{\mathrm{T}}\,) = K_1\, \boldsymbol{I_3}(\mathrm{m})\, h^{-\mathbf{1}}(\underline{\mathrm{T}})$

$$= \boldsymbol{I_3}(\mathbf{m}) / \textstyle\sum_{\mathbf{m=1}}^{\mathbf{n-1}} \boldsymbol{I_3}(\mathbf{m}) \quad (22)$$

where $\boldsymbol{I_3}(\mathbf{m}) = \int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta} \frac{a_2{}^{\beta}}{\Gamma\beta} A_1{}^{2\beta-1} \beta^{n}\, I_1(\mathrm{m},\beta) I_2(\mathrm{m},\beta)\, \mathrm{d}\beta$ (23)

where $I_1(\mathrm{m},\beta)$, $I_2(\mathrm{m},\beta)$ are same as in (16) and (17) respectively.

4. BAYES ESTIMATES UNDER ASYMMETRIC LOSS FUNCTION USING INFORMATIVE PRIOR:

In this section, we have obtained Bayes estimates of the change point and parameters θ_1 and θ_2. Here we have used a very useful asymmetric loss function known as the Linex Loss Function. It was introduced by Varian in 1975.

Minimizing the posterior expectation of the Linex loss function E_m [L_4 (m, d)], where E_m [L_4 (m, d)] denotes the expectation of L_4 (m, d) with respect to posterior density $g(\boldsymbol{m}\,|\underline{\mathrm{T}}\,)$. We get the Bayes estimate of 'm' by means of the nearest integer value, say $\mathrm{m}_{\mathrm{L}}^{*}$, using Linex Loss Function as under:

$$m_L^* = -\frac{1}{q_1} \ln[E(e^{-\mathrm{m}q_1})]$$

$$= -\frac{1}{q_1} \ln\left[\frac{\sum_{m=1}^{n-1} e^{-\mathrm{m}q_1} I_3(\mathrm{m})}{\sum_{m=1}^{n-1} I_3(\mathrm{m})}\right] \quad (21)$$

where $I_3(\mathrm{m})$ same as in (23).

Minimizing expected loss function $E_{\theta_1}[L_4\,(\theta_1, \mathrm{d})]$ and using posterior distribution (19) and we get the Bayes estimates of θ_1, using Linex loss function as

$$\theta_{1L}^{*} = -\frac{1}{q_1}\ln[E(e^{-\theta_1 q_1})]$$

$$= -\frac{1}{q_1}\ln[\int_0^\infty g_1(\theta_1|\underline{X}).\, e^{-\theta_1 q_1}\, d\theta_1]$$

$$= -\frac{1}{q_1}\ln[K_1 \sum_{m=1}^{n-1} \int_0^\infty [\theta_1{}^{2m-\beta-1} e^{-(\theta_1 A_2 + {a_1}/{\theta_1} + \theta_1 q_1)}\, d\theta_1] \qquad 2A_3{}^{-[2(n-m)-\beta]/2}$$

$$\left[\frac{1}{a_2}\right]^{-[2(n-m)-\beta]/2} \text{Bessel K}[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]$$

$$\int_{\beta_1}^{\beta_2} \frac{a_1{}^\beta}{\Gamma\beta}\frac{a_2{}^\beta}{\Gamma\beta} A_1{}^{2\beta-1}\beta^n\, d\beta h^{-1}(\underline{T})]$$

$$= -\frac{1}{q_1}\ln[K_1 \sum_{m=1}^{n-1} 2\{A_2 + q_1\}^{-[2m-\beta]/2} \left[\frac{1}{a_1}\right]^{-[2m-\beta]/2}$$

$$\text{Bessel K}\,[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2+q_1}\,]$$

$$2A_3{}^{-[2(n-m)-\beta]/2} \left[\frac{1}{a_2}\right]^{-[2(n-m)-\beta]/2} \text{Bessel K}[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]$$

$$\int_{\beta_1}^{\beta_2} \frac{a_1{}^\beta}{\Gamma\beta}\frac{a_2{}^\beta}{\Gamma\beta} A_1{}^{2\beta-1}\beta^n\, d\beta h^{-1}(\underline{T})] \qquad (22)$$

where Bessel $K[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2+q_1}]$ and

Bessel $K[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]$ are same as is in (18)

Minimizing expected loss function $E_{\theta_1}[L_4(\theta_2, d)]$ and using posterior distribution (20), we get the Bayes estimates of θ_2, using Linex Loss Function as

$$\theta_{2L}^{*} = -\frac{1}{q_1}\ln[E(e^{-\theta_2 q_1})]$$

$$= -\frac{1}{q_1}\ln[\int_0^\infty g_1(\theta_2|\underline{X}).\, e^{-\theta_2 q_1}\, d\theta_2]$$

$= -\frac{1}{q_1}\ln[K_1 \sum_{m=1}^{n-1} \int_0^\infty \theta_2^{2(n-m)-\beta-1} e^{-(\theta_2 A_3 + {a_2}/{\theta_2})} e^{-\theta_2 q_1}\, d\theta_2]$ $2A_2^{-[2m-\beta]/2}$

$\left[\frac{1}{a_1}\right]^{-[2m-\beta]/2}$ Bessel K $[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}]$

$\int_{\beta_1}^{\beta_2} \frac{a_1^{\beta}}{\Gamma\beta}\frac{a_2^{\beta}}{\Gamma\beta} A_1^{2\beta-1}\beta^n\, d\beta\, h^{-1}(\underline{T})]$

$= -\frac{1}{q_1}\ln[\, K_1 \sum_{m=1}^{n-1} 2\{A_3+q_1\}^{-[2(n-m)-\beta]/2}$

$\left[\frac{1}{a_2}\right]^{-[2(n-m)-\beta]/2}$ Bessel K$[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3+q_1}]$

$2A_2^{-[2m-\beta]/2}\left[\frac{1}{a_1}\right]^{-[2m-\beta]/2}$ Bessel K $[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}$

$\int_{\beta_1}^{\beta_2} \frac{a_1^{\beta}}{\Gamma\beta}\frac{a_2^{\beta}}{\Gamma\beta} A_1^{2\beta-1}\beta^n\, d\beta\, h^{-1}(\underline{T})]$ (29)

where Bessel K$[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}]$ and

Bessel K$[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3+q_1}]$ are same as is in (18)

Minimizing expected loss function $E_\beta[L_4(\beta, d)]$ and using posterior distribution (21), we get the Bayes estimates of β, using Linex Loss Function as,

$\beta^*_L = -\frac{1}{q_1}\ln[E(e^{-p_1 q_1})]$

$= -\frac{1}{q_1}\ln[\int_{\beta_1}^{\beta_2} g_1(\beta|\underline{X}).\, e^{-\beta q_1}\, d\beta]$

$= -\frac{1}{q_1}\ln[K_1 \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \frac{a_1^{\beta}}{\Gamma\beta}\frac{a_2^{\beta}}{\Gamma\beta} A_1^{2\beta-1}\beta^n\; e^{-\beta q_1}\, d\beta\, h^{-1}(\underline{T})\;]$ (30)

The Bayes estimate m_E^* of 'm' using General Entropy Loss Function is explained below. It was proposed by Calabria and Pulcini in 1994.

Minimizing the expectation $[E_m$ $[L_5$ (m, d)] and using posterior distribution, we get the Bayes estimate 'm' by means of the nearest integer value say m_E^* , using General Entropy Loss Function as under:

$$m_E^* = [E(\mathrm{m}^{-q_3})]^{-\frac{1}{q_3}}$$

$$= \left[\frac{\sum_{m=1}^{n-1} \mathrm{m}^{-q_3} I_3(\mathrm{m})}{\sum_{m=1}^{n-1} I_3(\mathrm{m})}\right]^{-\frac{1}{q_3}} \tag{31}$$

where $I_3(\mathrm{m})$ is same as in (23).

Further, minimizing the expectation $[E_{\theta_1}$ $[L_5$ $(\theta_1$,d)] and using posterior distribution (19), we get Bayes estimate of θ_1 using General Entropy Loss Function as,

$$\theta_{1E}^* = [E(\theta_1{}^{-q_3})]^{-\frac{1}{q_3}}$$

$$= [K_1 \sum_{m=1}^{n-1} \int_0^\infty [\theta_1{}^{2m-\beta-q_3-1} e^{-(\theta_1 A_2 + {a_1}/{\theta_1})} \, \mathrm{d}\theta_1 \qquad 2A_3{}^{-[2(n-m)-\beta]/2}$$

$$\left[\frac{1}{a_2}\right]^{-[2(n-m)-\beta]/2} \text{Bessel K}[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]$$

$$\int_{\beta_1}^{\beta_2} \frac{a_1{}^\beta}{\Gamma\beta} \frac{a_2{}^\beta}{\Gamma\beta} A_1{}^{2\beta-1} \beta^n \, \mathrm{d}\beta \, h^{-1}(\underline{\mathrm{T}}) \,]^{-\frac{1}{q_3}}$$

$$= [K_1 \sum_{m=1}^{n-1} 2A_2{}^{-[2m-\beta-q_3]/2} \left[\frac{1}{a_1}\right]^{-[2m-\beta-q_3]/2}$$

$$\left[\frac{1}{a_2}\right]^{-[2(n-m)-\beta]/2} \text{Bessel K}[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]$$

$$\int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta} A_1{}^{2\beta-1}\beta^n \ d\beta\, h^{-1}(\underline{T}) \,]^{-\frac{1}{q_3}} \qquad (32)$$

where Bessel K$[-2m+\beta-q_3, 2\sqrt{a_1}\sqrt{A_2}]$ and

Bessel K$[-2(n-m)+\beta, 2\sqrt{a_2}\sqrt{A_3}]$ are same as in (18).

Minimizing expectation $[E_{\theta_2}\ [L_5\ (\theta_2\ , d)]$ and using posterior distribution (20), we get Bayes estimate of θ_2 using General Entropy Loss Function as:

$$\theta^*_{2E} = [E(\theta_2{}^{-q_3})]^{-\frac{1}{q_3}}$$

$$= [K_1 \sum_{m=1}^{n-1} \int_0^{\infty} \theta_2{}^{2(n-m)-\beta-q_3-1} e^{-(\theta_2 A_3 + {}^{a_2}/_{\theta_2})}\, d\theta_2$$

$$2A_2{}^{-[2m-\beta]/2} \quad \left[\frac{1}{a_1}\right]^{-[2m-\beta]/2} \text{Bessel K}\,[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}\,]$$

$$\int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta} A_1{}^{2\beta-1}\beta^n\, e^{-\beta q_1}\, d\beta\, h^{-1}(\underline{T}) \,]^{-\frac{1}{q_3}}$$

$$= [\, K_1 \sum_{m=1}^{n-1} 2A_3{}^{-[2(n-m)-\beta-q_3]/2} \left[\frac{1}{a_2}\right]^{-[2(n-m)-\beta-q_3]/2}$$

$$\text{Bessel K}[-2(n-m)+\beta-q_3, 2\sqrt{a_2}\sqrt{A_3}]\ 2A_2{}^{-[2m-\beta]/2}$$

$$\left[\frac{1}{a_1}\right]^{-[2m-\beta]/2} \text{Bessel K}\,[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}\,]$$

$$\int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta} A_1{}^{2\beta-1}\beta^n\, e^{-\beta q_1}\, d\beta\, h^{-1}(\underline{T}) \,]^{-\frac{1}{q_3}} \qquad (33)$$

where Bessel K$[-2m+\beta, 2\sqrt{a_1}\sqrt{A_2}]$ and

Bessel K$[-2(n-m)+\beta-q_3, 2\sqrt{a_2}\sqrt{A_3}]$ are same as in (18).

Minimizing expectation $[E_\beta$ $[L_5$ $(\beta$, d)] and using posterior distributions (21), we get Bayes estimate of θ_β using General Entropy Loss Function as:

$$\beta^*_E = [E(\beta^{-q_3})]^{-\frac{1}{q_3}}$$

$$= [K_1 \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \frac{a_1^{\beta}}{\Gamma\beta}\frac{a_2^{\beta}}{\Gamma\beta} A_1^{2\beta-1} \beta^n \beta^{-q_3} d\beta\, h^{-1}(\underline{T})\ ]^{-\frac{1}{q_3}} \quad (34)$$

5. NUMERICAL STUDY:

We have generated 20 random observations from proposed Weighted Weibull Length Biased change point model. The first eight observations are with $\beta = 2.5$ and $\theta_1 = 0.005$ and next twelve are with $\beta = 2.5$ and $\theta_2 = 0.002$. Here, we note that θ_1 and θ_2 themselves were random observations from inverted gamma distributions with prior means $\mu_1 = 0.05$, $\mu_2 = 0.02$ and variance $\sigma_1^2 = 0.000$, σ_2^2=0.0008 resulting in $a_1 = 0.0075$ *and* $a_2 = 0.0030$. These observations are given in Table 1.

Table 1

Generated observations from proposed change point model

I	1	2	3	4	5	6	7	8	9	10
X_i	0.022	0.95	0.111	0.549	0.846	0.666	0.198	0.306	0.055	0.156
I	11	12	13	14	15	16	17	18	19	20
X_i	0.291	0.228	0.460	0.396	0.783	0.999	0.001	0.108	0.888	0.963

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Now, we have calculated the values of posterior mean of m, θ_1, θ_2, β. We have also calculated the posterior median and posterior mode of m. The results are shown below in Table 2.

Table 2

Prior Density	Bayes estimates of change point			Bayes estimates of Posterior means of parameters θ_1 and θ_2		Bayes estimates of Posterior means of parameters β
	Posterior Median	Posterior Mean	Posterior mode	θ_1	θ_2	β
Inverted Gamma prior	8.13	8.33	8.13	0.05	0.02	2.5

We also compute the Bayes estimates m_L^* , m_E^* of m, θ_{1L}^* , θ_{1E}^* of θ_1, θ_{2L}^* , θ_{2E}^* of θ_2 β_L^* , β_E^* of β, Using the results given in section 4 for the data given in table 3 and for different values of shape parameter $\boldsymbol{q_1}$ and $\boldsymbol{q_3}$,the results are shown in Tables 3 and Table 4.

TABLE 3 Bayes estimates using Linex Loss Function

Prior Density	q_1	m_L^*	θ_{1L}^*	θ_{2L}^*	β_L^*
Inverted	0.09	8	0.05	0.023	2.5
Gamma	0.10	8	0.05	0.022	2.5
prior	0.20	8	0.05	0.021	2.4
	1.2	7	0.03	0.018	2.2
	1.5	6	0.02	0.014	2.1
	-1.0	9	0.09	0.027	2.6
	-2.0	10	0.010	0.029	2.7

TABLE 4 Bayes estimates using General Entropy Loss Function

Prior Density	q_3	m_E^*	θ_{1E}^*	θ_{2E}^*	β_E^*
Inverted Gamma prior	0.09	8	0.05	0.023	2.5
	0.10	8	0.05	0.021	2.4
	0.20	8	0.05	0.020	2.3
	1.2	6	0.03	0.017	2.2
	1.5	5	0.02	0.015	2.0
	-1.0	9	0.09	0.025	2.6
	-2.0	10	0.10	0.028	2.8

Above table shows that for small values such as q_1= 0.09, 0.10, 0.20, Linex Loss Function is almost symmetric and nearly quadratic and the values of the Bayes Estimates under such a loss is not far from the posterior mean. Table 3 also shows that for q_1= 1.2, 1.5, Bayes Estimates are less than actual value of m=8.

For q_1= q_3= -1,-2, we can clearly see that the Bayes estimates are quite large than actual value m=8. It can be seen from the Table 3 and Table 4 that the negative sign of shape parameter of loss functions reflects under estimation is more serious than that over the estimation. Thus, problem of under estimation can be solved by taking the value of shape parameters of Linex and General Entropy Loss Functions as negative.

Table 4 shows that for small values of $|q_3|$, q_3= 0.09, 0.10, 0.20, the values of the Bayes estimate obtained using General Entropy Loss Functions are not far from the posterior mean. Table 4 also shows that for q_3= 1.2, 1.5, Bayes estimates are less than actual value of m=8.

Here, it is clearly seen from the Table 3 and Table 4 that positive sign of shape parameter of loss functions reflects over estimation is more serious than under estimation. Thus, problem of over estimation can be solved by taking the value of shape parameter of Linex and General Entropy Loss Functions as positive and high.

6. SENSITIVITY OF BAYES ESTIMATES

In this section, we have studied the sensitivity of the Bayes estimates obtained with respect to change in the prior of the parameter. The mean values μ_1 and μ_2 and variances$\sigma_1{}^2$ and $\sigma_2{}^2$ have been used as prior information in computing the parameters of the prior. Results are shown in Table 5.

Table 5

Posterior mean m* for the data given in Table 2

μ_1	μ_2	m*	m^*_E
0.005	0.005	8	8
0.005	0.006	8	8
0.005	0.008	8	8
0.007	0.002	8	8
0.002	0.002	8	8
0.4	0.002	8	8
0.002	0.004	8	8
0.003	0.005	8	8
0.004	0.006	8	8

Table 5 leads to the conclusion that m^* and m^*_E are robust with respect to the correct choice of the prior density of θ_1 (θ_2) and a wrong choice of the prior density of θ_1 (θ_2) Moreover, they are also robust with respect to the change in the shape parameter of General Entropy Loss Function.

7. CONCLUSIONS:

The results in all the tables lead to the conclusion that performance of posterior means has are closed to actual value of change point with correct choice of prior. 64% values of posterior median are closed to actual value of change point with correct choice of prior. 63% values of posterior mode are closed to actual value of change point with correct choice of prior. 67% values of m^*_L are closed to actual value of change point with correct choice of prior. 68% values of m^*_E are closed to actual value of change point with correct choice of prior.

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