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Bayesian Inference About the Estimation and Credible Intervals for A Continuous Distribution Under Bounded Loss Function

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ABSTRACT *In this paper, Bayes estimate of scale parameter of Gamma distribution is obtained using Informative prior distribution under a Bounded loss function, named Reflected gamma loss function and also Minimum Variance Unbiased estimator is obtained. Bayesian credible interval and Confidence interval is also constructed for the scale parameter of Gamma distribution. Comparisons are made between these estimators using Matlab software.*

KEYWORDS : Bayes estimator, Bounded loss function, Informative prior, Minimum variance unbiased estimator, Credible interval, Confidence interval

1. INTRODUCTION

The Gamma distribution plays an important role in many areas of the Statistics including areas of Life testing and Reliability. It is used to make realistic adjustment to exponential distribution in life testing situations. The fact that a sum of independent exponentially distributed random variables has a Gamma distribution, leads to the appearance of Gamma distribution in the theory of random counters and other topics associated with precipitation process. The Gamma distribution is a flexible distribution that commonly offers a good fit to any variable such as in Environmental, Meterology, Climatology and other physical situations. The Gamma distribution has also been used to model the size of insurance claims and rainfalls. This means that aggregate insurance claims and the amount of rainfall accumulated in a reservoir are modelled by a gamma distribution.

Suppose that X is a random variable drawn from Gamma distribution with the probability density function

$$f(x, \lambda) = \frac{e^{\frac{-x}{\lambda}} \; x^{\alpha-1}}{\Gamma\alpha \;\; \lambda^{\alpha}} \quad ; x > 0, \lambda > 0 ,\alpha > 0; \qquad (1.1)$$

where α is the shape parameter which is known and λ is the scale parameter which is unknown.

For a Bayesian analysis, loss function plays an important role and symmetric and asymmetric loss functions are used by most researchers. These loss functions are unbounded and widely employed in decision theory due to its elegant mathematical properties, not its applicability to the representation of a true loss structure (Leon and Wu, 1992). Various examples illustrate that in many situations, unbounded loss can be unduly restrictive and suggest that instead we should consider the properties of estimators based on a Bounded loss function. A bounded loss function avoids the potential explosion of the expected loss. Moreover, the nature of many decision problems and practical arguments require the use of Bounded loss functions, especially in Financial problems. For more details see Berger (1985). To overcome the shortcoming of unbounded loss, several bounded loss functions are proposed by many authors, for example, Spring (1993) proposed a bounded loss function named reflected normal loss function which is appropriate for estimation of location parameter and Towhidi and Behboodian (1999) proposed reflected gamma loss function which is appropriate for estimation of scale parameter and is given by

$$L(\hat{\lambda}, \lambda) = k\left\{1 - \exp\left[-q^2\left(\frac{\hat{\lambda}}{\lambda} - \log\frac{\hat{\lambda}}{\lambda} - 1\right)\right]\right\} \qquad (1.2)$$

where q > 0 is a shape parameter and k > 0 is the maximum loss parameter.

To see more about the discussion of bounded loss function, one can refer Bartholomew and Spiring (2002) and Kaminska (2010). Under reflected gamma loss function, Meghnatisi and Nematollahi (2009) studied the admissibility and inadmissibility of parameter of exponential distribution under various conditions. Yang, Zhou and Yuan (2013) studied Bayes estimation of parameter of Exponential distribution under Reflected Gamma loss function. In this paper, we will disuss MVUE and Bayes estimate of scale parameter and construct confidence interval and Bayesian credible interval for scale parameter. Properties of Bayes estimates of scale parameter will also discussed.

2. MINIMUM VARIANCE UNBIASED ESTIMATOR

Let X_1, X_2, X_3......................X_n be a random sample from a probability distribution having the probability density function f(x,λ) given by equation (1.1) and the joint density function of X_1, X_2......................X_n be

$L = \prod_{i=1}^{n} f(x_i, \lambda)$

To find an estimator $T = T(X_1, X_2$......................$X_n)$ of λ such that

E (T) = λ, and Var (T) = Minimum possible

is called the method of minimum variance and the statistic T is called the Minimum Variance Unbiased Estimator (MVUE) of λ.

To find T we minimize $E[T-\lambda]^2$ subject to the condition E (T) = λ .

The advantage of the method of minimum variance over other methods is that it gives also the variance of the estimator.

Here, Likelihood function is given by $L\left({x}/{\lambda}\right) = \dfrac{e^{\frac{-\sum_{i=1}^{n} x_i}{\lambda}}}{(\Gamma\alpha)^n \ \lambda^{n\alpha}} \prod_{i=1}^{n} {x_i}^{\alpha-1}$ (2.1)

$$\text{Log } L = \frac{-\sum x_i}{\lambda} + \log \prod_{i=1}^{n} {x_i}^{\alpha-1} - n \log \Gamma\alpha - n\,\alpha \log \lambda.$$

$$\frac{dlogL}{d\lambda} = \frac{\sum x_i}{\lambda^2} - \frac{n\alpha}{\lambda}$$

$$= \frac{\frac{\bar{x}}{\alpha} - \lambda}{\frac{\lambda^2}{n\alpha}}$$

Hence, MVUE of λ is $\hat{\lambda}_{MVUE} = \dfrac{\bar{x}}{\alpha}$ and variance of $\hat{\lambda}_{MVUE} = \dfrac{\lambda^2}{n\alpha}$.

3. BAYES ESTIMATION

The natural family of conjugate prior for λ is Inverted gamma (a,b), with p.d.f

$$g(\lambda) = \frac{e^{-a/\lambda} \ a^b}{\Gamma b \ \lambda^{b+1}} \ ; \lambda > 0, a > 0, b > 0 \qquad (3.1)$$

where b is shape parameter and a is scale parameter.

Applying Bayes theorem, we obtain from Equations (2.1) and (3.1), the Posterior density for λ as

$$P\left({\lambda}/{x}\right) = \frac{k \ e^{\frac{-\sum x_i + a}{\lambda}}}{\lambda^{n\alpha+b+1}} \ ; \lambda > 0, a > 0, b > 0, x > 0, \alpha > 0;$$

where k is independent of λ and $k^{-1} = \int_0^{\infty} \frac{e^{\frac{-\sum x_i + a}{\lambda}}}{\lambda^{n\alpha+b+1}} \, d\lambda$.

Therefore, Posterior density for λ is given by

$$P\left({}^{\lambda}\!/_{x} \right) = \frac{\sum x_i + a^{n\alpha+b}}{\Gamma(n\alpha+b)} \frac{e^{\frac{-(\sum x_i + a)}{\lambda}}}{\lambda^{n\alpha+b+1}} \tag{3.2}$$

Estimation of λ under Reflected Gamma Loss Function:

By using Reflected Gamma loss function

$$E\,[L\,(\hat{\lambda}, \lambda)] = \int_0^{\infty} L\,(\hat{\lambda}\,, \lambda\,) * P\left({}^{\lambda}\!/_{x} \right) d\lambda$$

$$= \int_0^{\infty} k\left\{1 - \exp\left[-q^2\left(\frac{\hat{\lambda}}{\lambda} - \log\frac{\hat{\lambda}}{\lambda} - 1\right)\right]\right\} * \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} \frac{e^{\frac{-(\sum x_i + a)}{\lambda}}}{\lambda^{n\alpha+b+1}} \, d\lambda$$

$$= \left\{ k \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} \int_0^{\infty} e^{\frac{-(\sum x_i + a)}{\lambda}} \frac{1}{\lambda^{n\alpha+b+1}} \, d\lambda \right\} - \{ k \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} \int_0^{\infty} e^{\frac{-(\sum x_i + a)}{\lambda}} \frac{1}{\lambda^{n\alpha+b+1}} e^{\left[-q^2\left(\frac{\hat{\lambda}}{\lambda} - \log\frac{\hat{\lambda}}{\lambda} - 1\right)\right]} d\lambda =$$

$$\left\{ k \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} \int_0^{\infty} e^{\frac{-(\sum x_i + a)}{\lambda}} \frac{1}{\lambda^{n\alpha+b+1}} \, d\lambda \right\} - \left\{ k \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} \int_0^{\infty} e^{\frac{-(\sum x_i + a)}{\lambda}} \frac{1}{\lambda^{n\alpha+b+1}} e^{\left[-q^2\left(\frac{\hat{\lambda}}{\lambda} - \log\frac{\hat{\lambda}}{\lambda} - 1\right)\right]} d\lambda \right\}$$

$$= k \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} * \frac{\Gamma(n\alpha+b)}{(\sum x_i + a)^{n\alpha+b}} - k \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} * e^{q^2} \int_0^{\infty} e^{\frac{-\hat{\lambda} q^2}{\lambda}} * \left(\frac{\hat{\lambda}}{\lambda}\right)^{q^2} * \frac{1}{\lambda^{n\alpha+b+1}} e^{\frac{-(\sum x_i + a)}{\lambda}} d\lambda$$

$$= k - k \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} e^{q^2} \hat{\lambda}^{q^2} \frac{\Gamma(n\alpha+b+q^2)}{(\sum x_i + a + q^2 \hat{\lambda})^{n\alpha+b+q^2}}$$

Now on solving $\frac{d}{d\lambda} E\,[L\,(\hat{\lambda}, \lambda)] = 0$, we obtain Bayes estimator of λ.

Thus, $\hat{\lambda}_{BL} = \frac{\sum x_i + a}{n\alpha + b}$

Theorem1: For a positive integer q, under Reflected gamma loss function, Bayes estimator of λ^q and λ^{-q} are given respectively by $\hat{\lambda}_{BL}^{q}$ and $\hat{\lambda}_{BL}^{-q}$, where

$$\hat{\lambda}_{BL}^{q} = \frac{\Gamma\,(n\alpha+b-q)}{\Gamma\,(n\alpha+b)} * \left(\sum x_i + a\right)^q \tag{3.3}$$

and

$$\hat{\lambda}_{BL}^{-q} = \frac{\Gamma\,(n\alpha+b+q)}{\Gamma\,(n\alpha+b)} * \left(\sum x_i + a\right)^{-q} \quad ; [q < n\alpha + b]. \tag{3.4}$$

Proof: For q > 0, the Bayes estimator of λ^q is

$$\hat{\lambda}^{q}_{BL} = E_{\lambda/x}\{\lambda^q\},$$

$$= \int_0^\infty \lambda^q \; P(\lambda/x)\, d\lambda$$

which on utilizing (3.2) gives

$$= \int_0^\infty \lambda^q \; \frac{(\sum x_i + a)^{n\alpha+b} \; e^{\frac{-\sum x_i + a}{\lambda}}}{\Gamma(n\alpha+b)\; \lambda^{n\alpha+b+1}} \; d\lambda$$

$$= \frac{\Gamma(n\alpha+b-q)}{\Gamma(n\alpha+b)} \quad (\sum x_i + a)^q$$

Hence, (3.3) holds. Similarly, we can prove (3.4).

4. **CREDIBLE INTERVAL FOR λ**

The Posterior mean is given by

$$E\left(\lambda/X\right) = \int_0^\infty \lambda \; P(\lambda/x)\; d\lambda$$

$$= \int_0^\infty \lambda \; \frac{(\sum x_i + a)^{n\alpha+b} \; e^{\frac{-\sum x_i + a}{\lambda}}}{\Gamma(n\alpha+b)\; \lambda^{n\alpha+b+1}} \; d\lambda$$

$$= \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} \int_0^\infty \lambda \; \frac{e^{\frac{-\sum x_i + a}{\lambda}}}{\lambda^{n\alpha+b+1}} \; d\lambda$$

$$= \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} * \frac{\Gamma(n\alpha+b-1)}{(\sum x_i + a)^{n\alpha+b-1}}$$

$$= \frac{\sum x_i + a}{n\alpha+b-1} \qquad (4.1)$$

$$E\left(\lambda^2/X\right) = \int_0^\infty \lambda^2 \; P(\lambda/x)\; d\lambda$$

$$= \int_0^\infty \lambda^2 \; \frac{(\sum x_i + a)^{n\alpha+b} \; e^{\frac{-\sum x_i + a}{\lambda}}}{\Gamma(n\alpha+b)\; \lambda^{n\alpha+b+1}} \; d\lambda$$

$$= \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} \int_0^\infty \lambda^2 \; \frac{e^{\frac{-\sum x_i + a}{\lambda}}}{\lambda^{n\alpha+b+1}} \; d\lambda$$

$$= \frac{(\sum x_i + a)^{n\alpha+b}}{\Gamma(n\alpha+b)} * \frac{\Gamma(n\alpha+b-2)}{(\sum x_i + a)^{n\alpha+b-2}}$$

$$= \frac{(\sum x_i + a)^2}{(n\alpha+b-1)\,(n\alpha+b-2)}$$

Posterior variance = $E\left(\lambda^2/_X\right) - \left[E\left(\lambda/_X\right)\right]^2$

$$= \frac{(\sum x_i+a)^2}{(n\alpha+b-1)\,(n\alpha+b-2)} - \left(\frac{\sum x_i+a}{n\alpha+b-1}\right)^2$$

$$= \frac{(\sum x_i+a)^2\,(\,(n\alpha+b-1) - (\sum x_i+a)^2\,\,(n\alpha+b-2)}{(n\alpha+b-1)^2\,\,(n\alpha+b-2)}$$

$$= \frac{(\sum x_i+a)^2}{(n\alpha+b-1)^2\,\,(n\alpha+b-2)} \qquad (4.2)$$

Bayesian Credible interval for λ is given by

$$m' \pm Z\alpha/_2\, s'$$

where m' is the posterior mean and s' is the posterior standard deviation

Hence Bayesian Credible interval for scale parameter of Gamma distribution is

$$\frac{\sum x_i+a}{n\alpha+b-1} \pm Z\alpha/_2 \quad \frac{\sum x_i+a}{n\alpha+b-1}\left(\frac{1}{(n\alpha+b-2)}\right)^{\frac{1}{2}} \qquad (4.3)$$

5. CONFIDENCE INTERVAL FOR λ

Confidence interval for λ is given by

$$\hat{\lambda}_{MVUE} \pm Z\alpha/_2\, S.E\,(\hat{\lambda}_{MVUE})$$

$$= \frac{\bar{x}}{\alpha} \pm Z\alpha/_2 \left(\frac{\lambda^2}{n\alpha}\right)^{\frac{1}{2}} \qquad (5.1)$$

Here, 95% Bayesian Credible interval (BCI) are also calculated along with Classical 95% confidence interval. The results are presented in table for different sample sizes.

Table 1:

Confidence Interval and Bayesian Credible Interval for different values of sample sizes and α = 1.5, a =1.5, b = 0.7

Sample Size n	Bayesian Credible Interval			Confidence Interval		
	Lower Limit	Upper Limit	Length of Interval	Lower Limit	Upper Limit	Length of Interval
30	1.0903	2.0448	0.9545	1.0873	2.2560	1.1687
60	2.3665	3.1208	0.7543	2.1340	2.9605	0.8265
90	2.2303	2.8495	0.6192	1.9813	2.6560	0.6747
120	1.9054	2.3761	0.4707	1.6374	2.2218	0.5844
150	1.9552	2.2756	0.3203	1.5854	2.1081	0.5227

6. SIMULATION STUDY

In order to assess the statistical performance of Bayes estimates and MVUE, a simulation study is conducted. The random samples are generated from (1.1) with true value of $\lambda = 2$ and $\alpha = 1, 1.5$ for different samples of sizes (n = 30, 60, 90, 120, 150). We use Matlab to generate these samples. All results are based on 1000 repetitions. Here, Bayes estimator and MVUE are computed under bounded loss functions for a = (0.4, 0.5, 1, 1.5). The Bayes estimator and MVUE for the scale parameter are averaged over the total number of repetitions. The results of the simulation study are summarized in tables 2 and 3. Graphs are plotted by taking values of $\hat{\lambda}_{BL}, \hat{\lambda}_{MVUE}$ along Y-axis and sample size along X-axis to see the behaviour of Bayes estimators and MVUE and to find an admissible estimator.

Table 2:

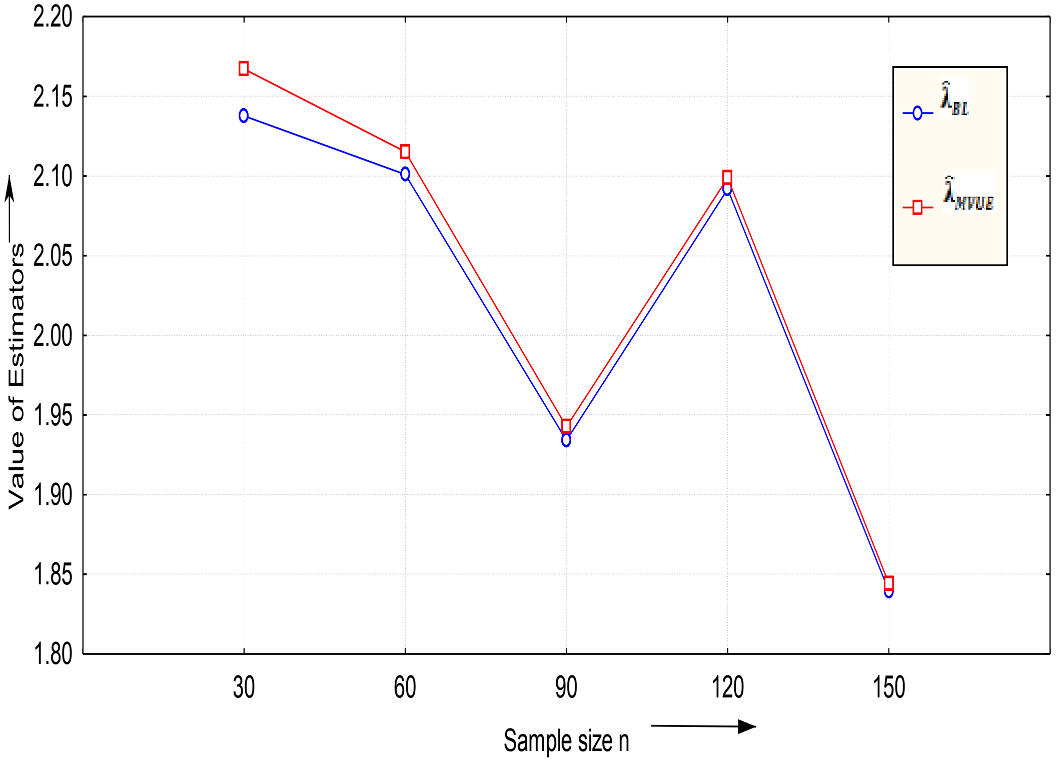
Values of Bayes estimator and minimum variance unbiased estimator of λ for different values of sample size, a and α .

Sample Size	$\alpha = 1, a = 0.4, b = 0.6$		$\alpha = 1, a = 0.5, b = 0.7$	
n	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{MVUE}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{MVUE}$
30	2.1377	2.1671	2.1661	2.2000
60	2.1009	2.1152	2.2836	2.3020
90	1.9341	1.9426	2.3516	2.3644
120	2.0917	2.0988	1.9312	1.9383
150	1.8396	1.8443	1.9821	1.9880

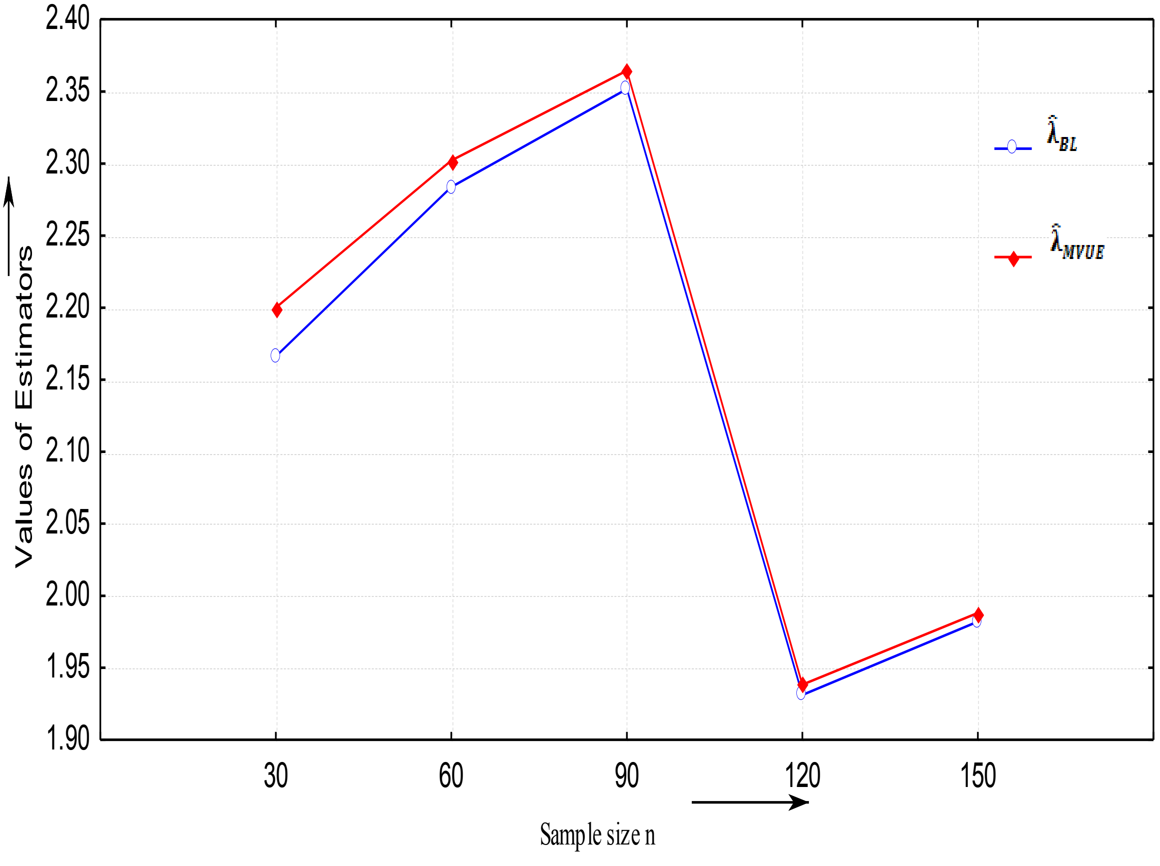
Table 3:

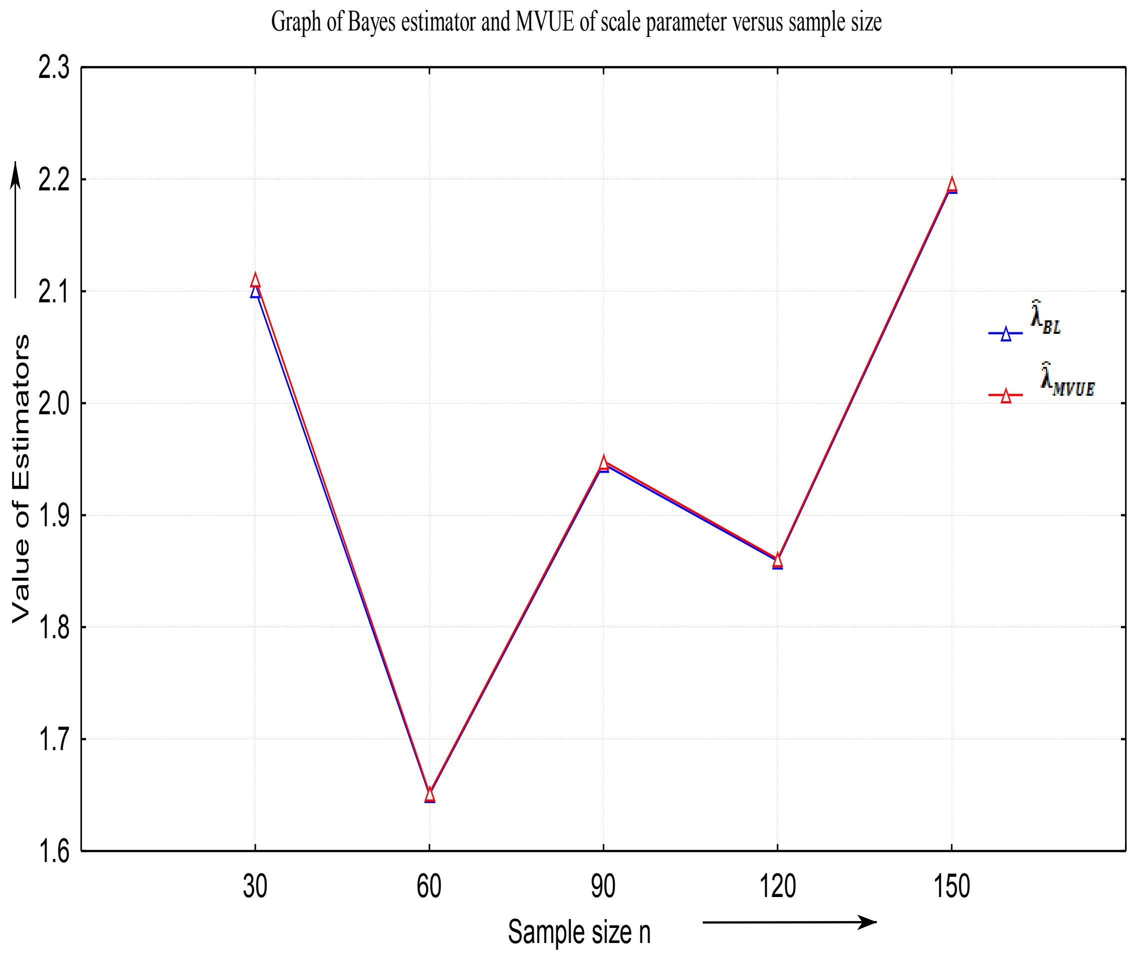
Sample Size	$\alpha = 1.5, a = 1, b = 0.7$		$\alpha = 1.5, a = 1.5, b = 0.7$	
n	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{MVUE}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{MVUE}$
30	2.1012	2.1112	2.1821	2.1824
60	1.6503	1.6520	2.1815	2.1817
90	1.9456	1.9482	1.8290	1.8278
120	1.8593	1.8610	1.9102	1.9095
150	2.1944	2.1967	2.2953	2.3000

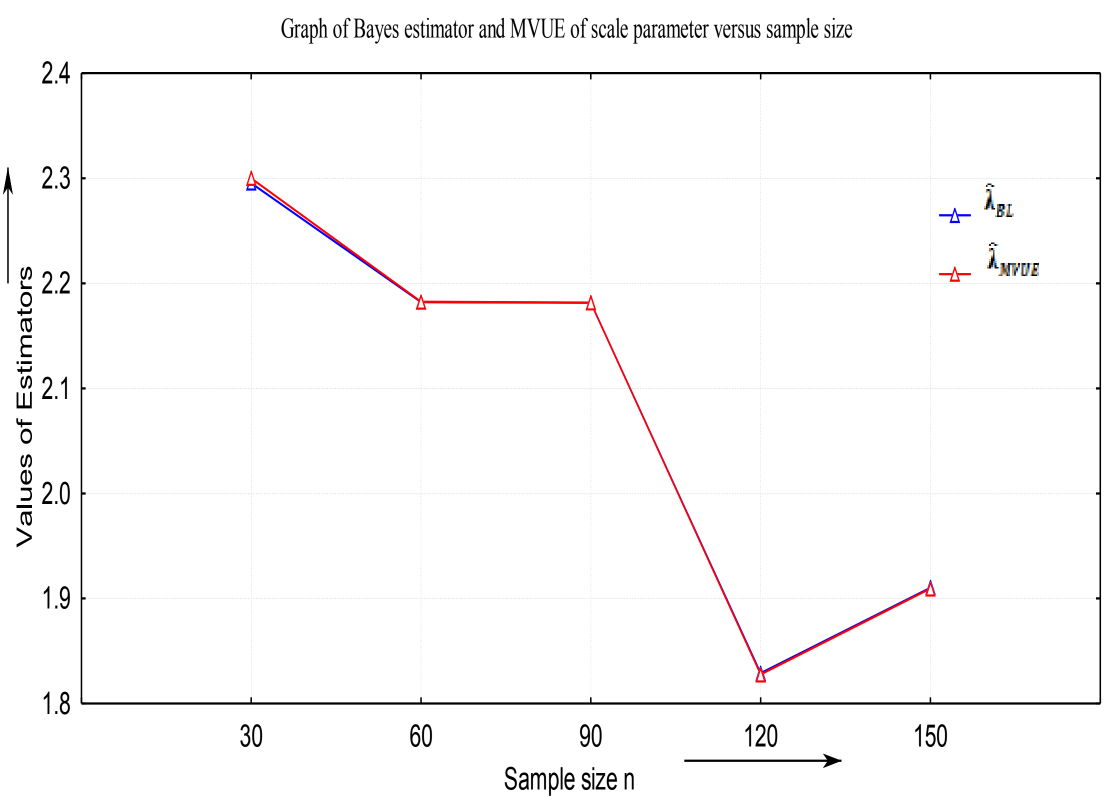
Graph of Bayes estimator and MVUE of scale parameter versus sample size



Graph of Bayes estimator and MVUE of scale parameter versus sample size




Graph of Bayes estimator and MVUE of scale parameter versus sample size
2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
Value of Estimators
30
60
90
120
150
Sample size n
$\hat{\lambda}_{BL}$
$\hat{\lambda}_{MVUE}$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2.0
1.9
1.8
Values of Estimators
30
60
90
120
150
Sample size n
$\hat{\lambda}_{BL}$
$\hat{\lambda}_{MVUE}$

CONCLUSION

From table 1, it is observed that the average length of confidence interval and Bayesian credible interval decreases when sample size increases. It is also noted from the table that average length of Bayesian Credible Interval is smaller than that of Confidence Interval.

From table 2-3, we conclude that in situations involving estimation of scale parameter, Bayes estimator under Bounded loss function could be effectively employed than MVUE as the convergence of the Bayes estimator towards the true values is more than that of MVUE. Thus, we suggest to use Bayes approach under bounded loss function for estimating scale parameter of Gamma distribution.

REFERENCES

1. Bartholomew, L. and F.A. Spiring, 2002. The inverted beta loss function: Properties and application. Transactions, 34: 1101-1109.
2. Berger, J.O., 1985. Statistical Decision Theory and Bayesian Analysis. 2nd Edn., Spring – Verlag, New York.
3. Chaturvedi, A. and Tomer, S.K. 2002. Classical and Bayesian reliability estimation of the negative binomial distribution. Jour. Applied Statist. Sci., 11(1), 33-43.
4. Kaminska, A., 2010. The equivalence of bayes and robust bayes estimators for various Loss functions. Stat. Papers, 51: 179-191.
5. Leon, R.W. and C.F. Wu, 1992. A theory of performance measures in parameter design. Stat. Sin., 2: 335-358.
6. Meghnatisi, Z. and N. Nematollahi, 2009. Inadmissibility of usual and mixed estimators of two ordered gamma scale parameters under reflected gamma loss function. J. Math. Extension, 3(2): 89-99.
7. Spring, F.A., 1993. The reflected normal loss function. Can. J. Stat., 21(3): 321-330.
8. Towhidi, M. and J. Behboodian, 1999. Estimation of scale parameter under a reflected gamma loss function. J. Sci. I. R. Iran, 10(4): 265-269.
9. Yang. L., Zhou. H., and Yuan. S., 2013. Bayes estimation of parameter of exponential distribution under a bounded loss function. Res. Math. Stat., 5(4): 28-31, 2013.