IF : 4.547 | IC Value 80.26

Volume-6, Issue-4, April - 2017 • ISSN No 2277 - 8160

Original Research Paper

Mathematics

STRUCTURE THEORMS

E.KIRAN BABU RESEARCH SCHOLAR, PP.MAT146, RAYALASEEMA UNIVERSITY, KURNOOL.

DR.B.RAJASEKHARA GOUD ASSOCIATE. PROFESSOR MATHEMATICS, G.PULLA REDDY ENGINEERING COLLEGE, (AUTONOMOUS) KURNOOL, A.P.518007 (INDIA)

ABSTRACT This paper deals with Structure of certain class of integers. Unitary Euler totient function, number theritacl functions play on important role in finding relations of the Structure theorems.

KEYWORDS : Residue system, Structure, Unitary systems.

INTRODUCTION :

The Paper studies three types of Residue systems and three types of Structure systems.

DEFINATIONS:

1. UNRRS: A Member X of CRS (Mod N) is member of Unitary Nagell Reduced Residue system modulo N relative to M [UNRRS (Mod N, M)] If$(X,N)^* = 1 = (M-X,N)^*$. The number of in a UNRRS (Mod N, M) is $\theta(M,N)$.

2. UJNRRS: An S-tuple $\bar{X}$ of JCRS (Mod N) **is** member of Unitary Jordan Nagell Reduce Residue system modulo N relative to $\bar{M}$ [UJNRRS (Mod N, $\bar{M}$) **]** If $\left((x_j),N\right)^* = \left((M_j - X_j),N\right)^* = 1$.

3. USNRRS: An S-tuple $< x+j >$ with X (Mod N) is member of Unitary Schemmel Nagell Reduced Residue system modulo N relative to $\bar{M}$ [USNRRS (Mod N, $\bar{M}$) **]** if $(x+j,N)^* = \left(M_j-(x+j),N\right)^* = 1$,for J=1, 2….s. The number of elements in [USNRRS (Mod N,$\bar{M}$)] is $\theta_s(\bar{M}$, N) .

THEOREM 1: (NagellStructure Theorem)

If d//n and b∈ $UNRRS(Mod\ d, M)$, then the numbers a≡ $b(Mod\ d)$ but Incongruent modulo N and $(a,N)^* = 1 = (M-a,N)^*$ are given by $a_k + t_k \frac{N}{d}$, where a_k runs through UNRRS (mod N/d, M) and t_k is the Solution of congruence $\lambda_k \frac{N}{d} \equiv b - a_k(mod\ d)$.

Proof: Let $(a,N)^* = 1 = (M-a,N)^*$ and a≡ $b(Mod\ d)$

Then $(a,d)^* = 1 = (M-a,d)^*$ and $(a,d)^* = (b,d)^*$; $(M-a,d)^* = (M-b,d)^*$

Further $\left(a,\frac{N}{d}\right)^* = 1 = \left(M-a,\frac{N}{d}\right)^*$.

Let a≡ $a_k\left(mod\frac{N}{d}\right)$ then $\left(a_k,\frac{N}{d}\right)^* = 1 = \left(M-a_k,\frac{N}{d}\right)^*$ write a = $a_k + \lambda_k\ \frac{N}{d}$.

Substituting this in equation a≡ $b(Mod\ d)$ **,** then $a_k + \lambda_k\ \frac{N}{d} \equiv b(mod\ d)$ or

$\lambda_k\ \frac{N}{d} \equiv (b-a_k)(mod\ d)$.

Since (d, N/d) =1, this has unique solution modulo d (say$t_k(mod\ d)$).

Conversely, let a = $a_k + t_k\ \frac{N}{d}$, where $\left(a_k,\frac{N}{d}\right)^* = 1 = \left(M-a_k,\frac{N}{d}\right)^*$.

Then clearly $(a,N)^* = 1 = (M-a,N)^*$ and a≡ $b(Mod\ d)$ also it is to be noted that The representation of a as a = $a_k + t_k\ \frac{N}{d}$ is unique modulo N.

Note: The number of integers belonging to UNRRS (Mod N, M) and congruent to b (mod d), $(b,d)^* = 1 = (M-b,d)^*$ and d//N is $\theta(M,N/d)$.

Example: Choose N=12. M= 6, d = 3 and b = 1.

Note that $(1,3)^* = 1 = (6-1,3)^*$.

Now $(a_k,4)^* = 1 = (6-a_k,4)^*$ => a_k = 1, 3, when $a_k = 1$, $t_k = 0$

$a_k = 3$, $t_k = 1$.

Therefore $a_k + t_k\ \frac{N}{d}$ = 1,7. observe that 1, 7≡ $1(mod\ 3)$ and 1,7 ∈ UNRRS(mod 12,6).

THEOREM 2: (Jordan -Nagell Stucture Theorem)

If d//N and $\bar{b}$ ∈ UJNRRS (Mod d, $\bar{M}$), then S- tuples $\bar{a} \equiv \bar{b}(mod\ d)$ but Incongruent modulo N and $\left((a_j),N\right)^* = \left((M_j-a_j),N\right)^* = 1$ are given by $\overline{a_k} + \overline{t_k}\ \frac{N}{d}$, $\overline{a_k}$ runs through UJNRRS (mod $\frac{N}{d}$, $\bar{M}$) and $\overline{t_k}$ is the solution of

The congruence $\overline{\lambda_k}\ \frac{N}{d} \equiv (\bar{b} - \overline{a_k})(mod\ d)$.

Proof: Let $\left((a_j),N\right)^* = 1 = \left((M_j - a_j),N\right)^*$ and $\bar{a} \equiv \bar{b}\ (mod\ d)$.

Then $\left((a_j),d\right)^* = 1 = \left((M_j - a_j),d\right)^*$ and $\left((a_j),d\right)^* = \left((b_j),d\right)^*$;

$\left((M_j - a_j),d\right)^* = \left((M_j - b_j),d\right)^*$

Further $\left((a_j),\frac{N}{d}\right)^* = 1 = \left((M_j - a_j),\frac{N}{d}\right)^*$.

Let $\bar{a} \equiv \overline{a_k}\ \left(mod\ \frac{N}{d}\right)$ then $\left(\left(a_{k_j}\right),\frac{N}{d}\right)^* = \left(\left(M_j - a_{k_j}\right),\frac{N}{d}\right)^* = 1$.

Write $\bar{a} = \overline{a_k} + \overline{\lambda_k}\ \frac{N}{d}$ Substitute this in the equation $\bar{a} \equiv \bar{b}\ (mod\ d)$, then

$\overline{a_k} + \overline{\lambda_k}\ \frac{N}{d} = \bar{b}\ (mod\ d)$ that is $\overline{\lambda_k}\ \frac{N}{d} \equiv \bar{b} - \overline{a_k}\ (mod\ d)$.

Since $\left(d,\frac{N}{d}\right) = 1$, this has unique solution modulo d say $\overline{t_k}\ (mod\ d)$.

Conversely, let $\overline{a_k}$ and $\overline{t_k}$ satisfy the following conditions.

$1 \le a_{k_j} \le \frac{N}{d}$,∀ j = 1, 2,……,s with $\left(\left(a_{k_j}\right),\frac{N}{d}\right)^* = 1 = \left(\left(M_j - a_{k_j}\right),\frac{N}{d}\right)^*$ and $\overline{t_k}$

Is the solution of the congruence (theorem 2).

Let $\bar{a} = \overline{a_k} + \overline{t_k}\ \frac{N}{d}$ then $\left((a_j),N\right)^* = 1 = \left((M_j - a_j),N\right)^*$ and $\bar{a} \equiv \bar{b}\ (mod\ d)$.

Further it can be noted that the representation of $\bar{a}$ is unique modulo N.

Note: The number of s-tuples belonging to UJNRRS (Mod N, $\overline{M}$) and congruent to $\bar{b}\ (mod\ d)$ Where $\left((b_j),d\right)^* = 1 = \left((M_j - b_j),d\right)^*$ and d//N is $\theta^{(s)}\left(\overline{M}\ ,\frac{N}{d}\right)$.

Example: Choose N = 6, s = 2, $\overline{M} = <7{,}10>$, $\bar{b} = <1{,}2>$ and d =3.

The 2-tuples $\bar{a} : \bar{a} \in$ UJNRRS (mod 6, < 7, 10 >) with $\bar{a} \equiv \bar{b}\ (mod\ 3)$ are < 1, 5> and < 4, 5 > which may also be expressed in the form $\overline{a_k} + \overline{t_k}\ \frac{N}{d}$ is stated in the above theorem. In fact $\overline{a_k}$ = < 1, 1 >, < 2, 1 > and the corresponding $\overline{t_k}$ = < 0, 2 >, < 1, 2>.

THEOREM 3: (Schemmel -Nagell Structure Theorem)

If d//N and < b+ j > ∈ USNRRS (Mod d, $\overline{M}$), then the distinct S- tuples of Consecutive integers, < a+ j> ≡ < b+ j > (mod d) belonging to USNRRS (mod N, $\overline{M}$) are given by < $a_k + t_k\ \frac{N}{d} + j$ >, where < a+ j> Runs through USNRRS (mod N/d,$\overline{M}$) and t_k is the solution of the congruence $\lambda_k\ \frac{N}{d} \equiv b - a_k (mod\ d)$.

Proof: Let < a+ j > ∈ USNRRS (Mod N, $\overline{M}$) and < a+ j> ≡ < b+ j > (mod d) Then $(a+j,d)^* = \left(M_j - (a+j),d\right)^*$ =1 and $\left(a+j,\frac{N}{d}\right)^* = \left(M_j - (a+j),\frac{N}{d}\right)^*$ =1

For each j= 1, 2…….s,\

Let $a \equiv a_k (mod\ N/d)$. Then $\left(a_k+j,\frac{N}{d}\right)^* = \left(M_j - (a_k+j),\frac{N}{d}\right)^*$ =1 ∀ j=1,2….s

Write a = $a_k + \lambda_k\ \frac{N}{d}$ the equation < a+ j> ≡ < b+ j > (mod d) yields

$a_k + \lambda_k\ \frac{N}{d} \equiv$ b (mod d) that is $\lambda_k\ \frac{N}{d} \equiv b - a_k$ (mod d) since (d, N/d) =1,

This has unique solution modulo d say t_k (mod d).

Conversely, Let a = Therefore $a_k + t_k\ \frac{N}{d}$,

Where < $a_k + j$ > ∈ USNRRS (Mod $\frac{N}{d}$, $\overline{M}$) and t_k is the solution of the congruence.

Then $(a+j,N)^* = \left(M_j - (a+j),N\right)^*$ =1 and < a+ j> ≡ < b+ j > (mod d) ∀ j=1,2….s .

Also it can be observed that the representation of a as a = $a_k + t_k\ \frac{N}{d}$ is Unique modulo N.

Note: The number of s- tuples of consecutive integers belonging to USNRRS (Mod N, $\overline{M}$) And congruent to < b+ j > (mod d), where < b+ j > ∈ USNRRS (Mod d, $\overline{M}$) and d//N is $\theta_s\left(\overline{M},\frac{N}{d}\right)$.

Example: Choose N = 30, s = 3, $\overline{M} = <9{,}10{,}11>$, b = 0 and d = 5

Now < $a_k + j$ > ∈ USNRRS (Mod 6,< 9,10,11>)

⇨ $a_k = 0{,}1$. When $a_k = 0$, $t_k = 0$

$a_k = 1$, $t_k = 4$

Therefore $a_k + t_k\ \frac{N}{d}$ $a_k + t_k\ \frac{N}{d}$ = 0, 25.

Note : <0+ j> ,< 25 + j> ∈ USNRRS (Mod 30,< 9,10,11>) and <0+ j> ,< 25 + j > ≡ < 0+ j > (mod 5) .

References:

1. T.M .Apostol: Arithmetical Properties of generalized Ramanujan sums, pacific J.math.41 (1972) 281-293.
2. E.Cohen: A class of arithmetical functions, Proc.Nat, Acad.sci.U.S.A. 41(1955) 939-944.
3. P.Vijaya sekhar reddy: Structure of a class of a integers and its application to Totient & Ramanujan sums.