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An Analysis of Numerical Solutions and Errors with Euler's Method

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ABSTRACT This paper mainly presents Euler method for solving initial value problems (IVP) for ordinary differential equations (ODE). The proposed method is efficient and practically well suited for solving these problems. In order to verify the accuracy, we compare numerical solutions with the exact solutions. The numerical solutions are in good agreement with the exact solutions. In order to achieve higher accuracy in the solution, the step size needs to be very small. Finally we investigate and compute the errors of the euler's method for different step sizes .

KEYWORDS : ODE, IVP, Euler method

INTRODUCTION

A differential equation is an equation involving independent variables, dependent variables and the derivatives of dependent variables with respect to independent variables.

The order of a differential equation is the order of the highest derivative appearing in the equation whereas its degree is the degree of the highest order derivative in the equation after removing fractional and negative exponents of the derivatives. The ODE is linear if no product of the dependent variable with itself or any of its derivatives occur in the equation, otherwise it is non linear. The solutions to the nth order equation depend on n parameters. To find these parameters, n conditions must be given. If these n conditions are given at a single point only, then the differential equation together with the conditions is called IVP of nth order. When the n conditions are given at two or more points, then the problem is called boundary value problem (BVP).

Numerical methods are generally used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Only a limited number of differential equations can be solved analytically. There are many analytical methods for finding the solution of ordinary differential equations. Even then there exist a large number of ordinary differential equations whose solutions cannot be obtained in closed form by using well-known analytical methods, where we have to use the numerical methods to get the approximate solution of a differential equation under the prescribed initial condition or conditions. There are many types of practical numerical methods for solving initial value problems for ordinary differential equations.
In this paper we present Euler Method for solving IVP of ODE
.

From the literature review we may realize that several works in numerical solutions of initial value problems using Euler method has been carried out. Many authors have attempted to solve IVP to obtain high accuracy rapidly by using numerous methods, such as Euler method. [1]-[13] also studied numerical solutions of initial value problems for ordinary differential equations using Euler method and other various numerical methods.

EULER'S METHOD

The slope at the beginning of the interval is taken as an approximation of the average slope over the whole interval. This approach, called *Euler's Method.*

Though in principle it is possible to use Taylor's method of any order for the given IVP to get good approximations, it has few draw backs like the scheme assumes the existence of all higher order derivatives for the given function **f(x,y)** which is not a requirement for the existence of the solution for any first order initial value problem. Even the existence of these higher derivatives is guaranteed it may not be easy to compute them for any given **f(x,y).** Because of the usage of higher order derivatives in the formula it is not convenient to write computer programs , that is the method is more suited for hand calculations. To overcome these difficulties, Euler developed scheme by approximating **y'** in the given IVP.

The derivative term in the first order IVP **y' = f(x, y) , y(x0) = y0**
Now we use Euler's method to numerically integrate Eq.

$$\frac{dy}{dx} = 5x^4 + 6.4\,x^3 + 12x^2 + 2x + 6.5$$

from x = 0 to x= 3 with a step size of **0.5**. The initial condition at x=0 is y = 1. the exact solution is given by equation

$$y = x^5 + 1.6x^4 + 4\,x^3 + x^2 + 6.5x + 1$$

The true solution and solution at x=0.5 is 5.13125 and 4.25 And, the error is
Error = true solution – approximate solution

= 5.13125 – 4.25=0.88125

or, expressed as percentage relative error, *Ep,* = +17.2%.

For the second step, the true solution and solution by euler's method at x=1.0 has been find out. The computation is repeated, and the results are compiled in Table shown below.

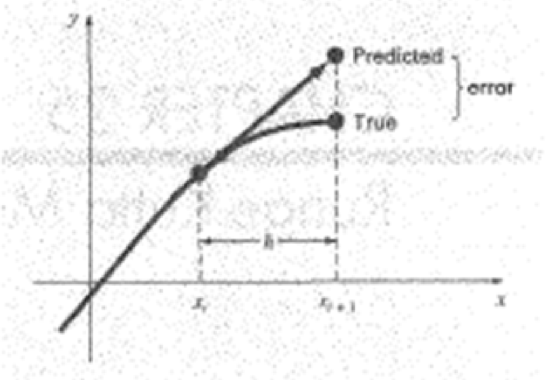
x	y_{true}	y_{Euler}	Percentage relative error(E_p)
0	1.00000	1.00000	
0.5	5.13125	4.25	+17.2
1.0	15.10000	10.05625	+33.4
1.5	42.19375	26.00625	+38.4
2.0	107.60000	67.7125	+37.1
2.5	246.15625	162.5625	+34.0
3.0	510.10000	353.46875	+30.7

Table: Comparison of true and approximate values of the integral of $y'=5x^4+6.4x^3+12x^2+2x+6.5$ with the initial condition that ***y=1 at x=0***. The approximate values were computed using Euler's method with a step size of 0.5.

ERROR ANALYSIS FOR EULER'S METHOD

Numerical solutions of ODEs involves two types of error: **Truncation error** and **Round-off error.** In numerical analysis and scientific computing, *truncation error* is the error made by truncating an infinite sum and approximating it by a finite sum. *Round-off* errors caused by the limited numbers of significant digits that can be retained by a computer.

The truncation errors are composed of two parts. The first is a *local truncation error* that results from an application of the method in question over a single step. The second is a *propagated truncation error* that results from the approximations produced during the previous steps. The sum of the two is the total, or *global truncation, error.*



Euler's method corresponds to the Taylor series up to and including the term **f(x,y)*h**. The comparison indicates that a truncation error occurs because we approximate the true solution using a finite number of terms from the Taylor series. We thus truncate, or leave out, a part of the true solution. For example, the truncation error in Euler's method is attributable to the remaining terms in the Taylor series expansion that were not included . The Taylor series provides only an estimate of the local truncation error-that is, the error created during a single step of the method. It does not provide a measure of the propagated and, hence, the global truncation error. In actual problems we usually deal with functions that are more complicated than simple polynomials. Consequently, the derivatives that are needed to evaluate the Taylor series expansion would not always be easy to obtain. Although these limitations preclude exact error analysis for most practical problems, the Taylor series still provides valuable insight into the behavior of Euler's method. we see that the local error is proportional to the square of the step size and the first derivative of the differential equation. These observations lead to some useful conclusions:

CONCLUSION

Generally, the approximation gets less accurate the further you are away from the initial value. Better accuracy is achieved when the points in the approximation are closer together. Your approximation is going to be above the actual curve if the function is *concave down* and below the actual curve if the function is *concave up* .The error can be reduced by decreasing the step size. The method will provide error-free predictions if the solution of the differential equation is linear, because for a straight line the second derivative would be zero. This latter conclusion makes intuitive sense because Euler's method uses straight-line segments to approximate the solution. Hence, Euler's method is referred to as a first-order method.

REFERENCES

[1] Abraham O., Improving the modified Euler method, Leonardo J. Sci, 2007, 10, p. 1-8.
[2] Islam, Md.A. , Accuracy Analysis of Numerical solutions of Initial Value Problems (IVP) for Ordinary Differential Equations (ODE). IOSR Journal of Mathematics, 11, 18-23.
[3] Hall, G. and Watt, J.M. (1976) Modern Numerical Methods for Ordinary Differential Equations. Oxford University Press, Oxford.
[4] Butcher J. C., Numerical methods for ordinary differential equations in the 20th, Century, J. Comput. Appl. Math., 2000, 125, p. 1-29.
[5] Ogunrinde, R.B., Fadugba, S.E. and Okunlola, J.T. (2012) On Some Numerical Methods for Solving Initial Value Problems in Ordinary Differential Equations. IOSR Journal of Mathematics, 1, 25-31.
[6] Akanbi, M.A. (2010) Propagation of Errors in Euler Method, Scholars Research Library. Archives of Applied Science Research, 2, 457-469.
[7] Euler H., Institutiones calculi integralis, Volumen Primum (1768), Opera Omnia, Vol. XI, B. G. Teubneri Lipsiae et Berolini, MCMXIII, 1768.
[8] Butcher J. C., General linear methods: A survey, Appl. Numer. Math., 1985, 1. 107 – 108.
[9] Fatunla S. O., Numerical methods for initial value problems in ordinary differential equations, Academic Press Inc. (London) Ltd. p. 48-51.
[10] Abraham O., A fifth-order six stage explicit Runge-Kutta method for the solution of initial value problems, M. Tech Dissertation, Federal University of Technology, Minna, Nigeria, 2004.
[11] Hull T. E., Enright, W. H., Fellen, B. M., and Sedgwick, A. E., Comparing numerical methods for ordinary differential equations, SIAM J. Numer. Anal., 1972, 9, 603 – 637.
[12] Hossain, Md.S., Bhattacharjee, P.K. and Hossain, Md.E. (2013) Numerical Analysis. Titas Publications, Dhaka.
[13] Burden, R.L. and Faires, J.D. (2002) Numerical Analysis. Bangalore, India.