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SEMIGROUP VARIETIES AND ITS GEOMETRIC INTERPRETATION

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ABSTRACT Our main aim is to derive the embedding properties of modular lattice within its variety, into an algebraic Lattice. We extend here that every lattice which is either algebraic modular spatial or bi-algebraic is strongly spatial. Hermann and Roddy derived that every modular lattice embeds into some algebraic and spatial lattice. We show here that every n-distributive lattice embeds within its variety. It is illustrated by an example that only those lattice with a least and greatest element can be embedded which is join semi-distributive. The main derivation is that for every positive integer n, every n-distributive lattice is embedded within its variety which generalises word problem derived by C. Herrmann. Herrmann, Pickering, and Roddy, proved that every modular lattice L embeds into some algebraic and spatial lattice $\bar{L}$ that satisfies the same identities as L.

KEYWORDS : Semigroup, Lattice, Irreducible, Spatial lattice, vector space

INTRODUCTION

The geometric description provided for n-distributive lattices that is generated by its finite members, thus has a computable word problem.s R. Frecs results for modular lattices do not hold G. Herrmann was shown that while every modular lattice embeds into some algebraic, modular, and spatial lattice But for n-distribution lattice it is mere a chance Herrmann and Roddy developed at the axiomatization of the abstract projective geometries associated with algebraic spatial lattices in particular the so-called *Triangle Axiom*, shows that the existential quantifier involved in that axiom prevents us from expressing an infinite projective space as a "limit" of finite projective spaces in any satisfactory way. We have shown here that it does not appear in the case of n-distributive lattices.

Hutchinson and Czedli characterize those rings R for which the word problem for free lattices in the variety generated by all subspace lattices of left R-`modules is decidable. This class of rings includes all fields, and also the ring Z of all integers as well as its quotient rings Z/mZ for positive integers m.

Herrmann and Huhn proved that the word problem in the variety generated by all complemented modular lattices is solvable. In case of results derived by Goodearl about von Neumann regular rings It has been proved that the variety generated by complemented Arguesian lattices with an extra unary operation symbol for complementation is generated by its finite members, and thus that the word problem for free lattices with complementation in the variety generated by those structures is decidable. It is extended to the variety generated by all complemented modular lattices with a unary operation symbol for complementation, where residual finiteness is replaced by residual finite length.

An element p in a lattice L is completely join-irreducible, if there is a largest element smaller than p. L is spatial if every element of L is a join of points. In such a case, elements of L are identified with certain sets of points of L. If, in addition, L is algebraic, then it gives a geometric description of L. In case of equational properties of lattices, the geometric description enables to prove representation results. we prove here that every lattice which is either well-founded or bi-algebraic is strongly spatial, which extends the result, that not every lattice can be enclosed into some bi-algebraic lattice. We put the following notations for convenience which slightly differs from earlier used notions due to Birkhhof.

$$Q{\downarrow}X = \{q \epsilon\ Q \mid (\exists x \epsilon\ X)(q \leq x)\},$$

$$Q{\uparrow}X = \{q \epsilon\ Q \mid (\exists x \epsilon\ X)(q \geq x)\},$$

$$Q{\downarrow\downarrow}X = \{q \epsilon\ Q \mid (\exists x \epsilon\ X)(q < x)\},$$

for all subsets X and Q in a poset P. Let us consider $Q{\downarrow}a := Q{\downarrow}\{a\}$, $Q{\uparrow}a := Q{\uparrow}\{a\}$, and $Q{\downarrow\downarrow}a := Q{\downarrow\downarrow}\{a\}$, for each $a \in P$. A subset X of P is a lower subset of P if $X = P{\downarrow}X$. A subset X of P refines a subset Y of P, defined by $X \leq_{ref} Y$, if $X \subseteq P{\downarrow}Y$. where $x \ll y$ denotes the "way-below" relation, $X \ll Y$ for the refinement relation on subsets. We write $X <_{ref} Y$ for the conjunction of $X \leq_{ref} Y$ and $X \neq Y$.

Classification of lattice – structure

For elements x and y in a poset P, let x . y hold ("x is a lower cover of y", or "y is an upper cover of x") if $x < y$ and there is no $z \in P$ such that $x < z < y$. An element p in a join-semilattice L satisfies the following relations.

(i) It is join-irreducible if $p = \vee X$ implies that $p \in X$, for any finite subset X of L;

(ii) It is completely join-irreducible, if the set of all elements smaller than p has a largest element, then denoted by p_*. Hence, every point is join-irreducible;

(iii) an atom of L if L has a zero element and 0 ≻ p.

Let us denote by J(L) ($J_c(L)$, At(L), respectively) the set of all join-irreducible elements (points, atoms, respectively) of L. At(L) ⊆ $J_c(L)$ ⊆ J(L). A subset Σ of L is join-dense in L if every element of L is a join of elements of Σ. Equivalently, for all a, b ∈ L with a ≰ b, there exists x ∈ Σ such that x ≤ a and x ≰ b. An element a in L is compact if for every nonempty directed subset D of L with a join, a ≤ ⋁D implies that a ∈ P ↓ D. L is said to be

(iv) spatial if the set of all points of L is join-dense in L;

(v) atomistic if the set of all atoms of L is join-dense in L;

(vi) compactly generated if the set of all compact elements of L is join-dense in L;

(vii) algebraic if it is complete and compactly generated;

(viii) bi-algebraic if it is both algebraic and dually algebraic.

A lattice L is upper continuous if the equality a ∧ WD = W(a ∧ D) (where a ∧ D := {a ∧ x | x ∈ D}) holds for every a ∈ L and every nonempty directed subset D of L with a join.

Theorem 1
Every compactly generated lattice L is upper continuous and every point of L is compact.

Proof
Let a ∈ L and let D be a nonempty directed subset of L with a join, we first prove that a ∧ ⋁D ≤ ⋁(a ∧ D). Let c ∈ L compact with c ≤ a ∧ ⋁D. As c ≤ ⋁D and c is compact, there exists d ∈ D such that c ≤ d. we thus find that c ≤ a ∧ d ≤ ⋁(a ∧ D). But L is compactly generated, the upper continuity of L is an immediate consequence of the hypothesis. Let p be a point of L and let D be a directed subset of L with a join such that p ≤ ⋁D. If p /∈ L ↓ D, then p ∧ x ≤ p. for each x ∈ D, By suitable application of the upper continuity of L, we get p = p ∧ ⋁D ≤ p_*, which gives a contradiction. Hence, the theorem is proved.

Theorem 2
Every algebraic atomistic lattice is strongly spatial.

Proof
The points of an algebraic atomistic lattice L are exactly its atoms. Let us consider the case when A is a finite cover of an atom p of L, By using the compactness of p together with the fact that each element of A is a join of atoms that there exists a finite cover X of p, consisting only of atoms, refining A. But every irredundant cover Y of p contained in X refines A and belongs to mCov(p). Let us illustrate by an example that the non-modular case, algebraic and spatial does not imply strongly spatial. Let $\omega^\partial := \{n^* \mid n < \omega\}$ with $0^* > 1^* > 2^* > \cdots$. Then the lattice

$$L := \omega^\partial \cup \{0, c\},$$

with the only new relations $0 < c < 0^*$ and $0 < n^*$ for each n < ω, is algebraic and spatial although not strongly spatial. It is thus find that it that this is 2-distributive, and not dually algebraic.

Theorem 3
Let p be an element in a complete, lower continuous lattice L. If either p is countably compact or L is either well-founded or dually well-founded, then every join-cover of p can be refined to some minimal join-cover of p.

Proof
Let C ∈ Cov(p), and C to an element of mCov(p). we prove here that tCov(p), endowed with the refinement order, is well-founded. If not possible there exists a sequence $\vec{A} = (A_n \mid n < \omega)$ from tCov(p) such that the inequality $A_{n+1} <_{ref} A_n$ holds for each n < ω.

An element x ∈ L is $\vec{A}$-*reducible* if there exists a natural number k such that $|A_k \downarrow x| \geq 2$. Hence, the sequence $(|A_k \downarrow x| \mid k < \omega)$ is nondecreasing, thus the $\vec{A}$-reducibility of x is equivalent to assertion that $|A_k \downarrow x| \geq 2$ for all large enough k < ω. Let us choose

Bn := { x ∈ A_n | x is $\vec{A}$-reducible}, for each n < ω .

$A_n \downarrow x = \{x\}$ for all m ≤ n < ω and all $x \in A_m \setminus B_m$

In particular, if $B_n = \emptyset$, then $A_n \subseteq A_{n+1}$, thus $A_n = A_{n+1}$ as each of these sets covers p tightly, which contradicts the assumption that $A_{n+1} <_{ref} A_n$. Therefore, B_n is nonempty.

Also, for each n < ω, there exists k > n such that $|A_k \downarrow x| \geq 2$ for each $x \in B_n$. we get a strictly increasing sequence $(n_i \mid i < \omega)$ of natural numbers, with $n_0 = 0$, such that for all i < ω and all $x \in B_{ni}$, the set $A_{ni+1} \downarrow x$ has at least two elements. Set $\vec{A}' := (A_{ni} \mid i < \omega)$. Hence, the notions of $\vec{A}$-reducibility and $\vec{A}'$-reducibility are equivalent, By replacing $\vec{A}$ by $\vec{A}'$ and assuming that $n_i = i$ for each i < ω. We obtain

For all m < n < ω and for all $x \in B_m$, $|An \downarrow x| \geq 2$

In case $B_n \cap A_{n+1} = \emptyset$ and B_{n+1} refines B_n, for each n < ω.

Let $u \in B_n \cap A_{n+1}$. We find that $A_{n+1} \downarrow u$ covers u tightly, but $u \in A_{n+1}$, thus $A_{n+1} \downarrow u = \{u\}$, which contradicts the assumption. Hence $B_n \cap A_{n+1} = \emptyset$.

Let $v \in B_{n+1}$. There exists $u \in A_n$ such that v ≤ u. If u /∈ B_n, then, $A_{n+1} \downarrow u = \{u\}$, thus v = u, a contradiction as u is $\vec{A}$-irreducible while v is $\vec{A}$-reducible; so $u \in B_n$. Hence, the theorem is proved.

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