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A New Class Of Vague Submodule

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ABSTRACT In this paper, we define a new class of vague submodule and obtain several properties of vague submodule of a module. Further we derive the necessary and sufficient conditions for vague submodules.

KEYWORDS : Vague set, Vague submodule of a module.

1.Introduction: Algebraic structures play a vital role in mathematics and numerous applications of these structures are seen in many disciplines such as computer sciences, information sciences, theoretical physics, and so on. This inspires research in various concepts of abstract algebras in fuzzy sets. There are number of generalizations of Zadeh's fuzzy[9] set theory so for reported in the literature viz., i-fuzzy theory, two-fold fuzzy theory, etc. Zhao Jianli [10] introduced and examined the notion of a fuzzy module over a ring in 1993. The notion of fuzzy groups defined by Rosen field [8] is the first application of fuzzy set theory in algebra. The concept of vague set theory was introduced by Gau and Buehrer[4] in 1993. Ranjit Biswas[7] introduced the study of vague algebra by studying vague groups. AmarendraBabu and Ramarao [1,2] has also introduced the notion of vague additive groups, vague rings and vague fields. The objective of this paper is to study the concept of vague submodule of a module and its related properties.

2.Preliminaries

Definition 2.1: [3] A vague set A in the universe of discourse U is characterized by two membership functions given by:

(i) A true membership function $t_A : U \to [0,1]$ and
(ii) A false membership function $f_A : U \to [0,1]$

Where $t_A(x)$ is a lower bound on the grade of membership of x derived from the "evidence for x", $f_A(x)$ is a lower bound on the negation of x derived from the "evidence for x", and $t_A(x) + f_A(x) \le 1$ Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(x), 1 - f_A(x)]$ of [0,1]. This indicates that if the actual grade of membership of x is μ(x), then, $t_A(x) \le \mu(x) \le 1 - f_A(x)$ The vague set A is written as $A = \{\langle x, [t_A(x), 1 - f_A(x)] \rangle / u \in U\}$ where the interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A, denoted by $V_A(x)$.

Definition 2.2:[4] Let A and B be VSs of the form $A = \{\langle x, [t_A(x), 1 - f_A(x)] \rangle / x \in X\}$ and $B = \{\langle x, [t_B(x), 1 - f_B(x)] \rangle / x \in X\}$ Then

(i) $A \subseteq B$ if and only if $t_A(x) \le t_B(x)$ and $1 - f_A(x) \le 1 - f_B(x)$ for all x∈X
(ii) A=B if and only if $A \subseteq B$ and $B \subseteq A$
(iii) $A^c = \{\langle x, f_A(x), 1 - t_A(x) \rangle / x \in X\}$
(iv) $A \cap B = \{\langle x, \min(t_A(x), t_B(x)), \min(1 - f_A(x), 1 - f_B(x)) \rangle / x \in X\}$
(v) $A \cup B = \{\langle x, (t_A(x) \vee t_B(x)), (1 - f_A(x) \vee 1 - f_B(x)) \rangle / x \in X\}$

For the sake of simplicity, we shall use the notation $A = \langle x, t_A, 1 - f_A \rangle$ instead of $A = \{\langle x, [t_A(x), 1 - f_A(x)] \rangle / x \in X\}$

Definition 2.3:[6] A vague set A on X is called a vague subalgebra of x if, for any $x \in X$, we have $t_A(xy) \ge \min\{t_A(x), t_A(y)\}$ and $1 - f_A(xy) \ge \min\{1 - f_A(x), 1 - f_A(y)\}$

Definition 2.4:[1] Let X be a ring and R be a vague set of X. Then R is a vague ring of X if the following conditions are satisfied:

1. $V_R(x+y) \le i\max\{V_R(x), V_R(y)\}$,
2. $V_R(-x) \le \{V_R(x)$,
3. $V_R(xy) \ge i\min\{V_R(x), V_R(y)\}$ *for all* $x, y \in X$

Definition 2.5:[5] Let f be a mapping from a set X into set Y. Let B be a vague set in Y. Then the inverse image of B i.e. f^{-1} [B], is the vague set in X by $V_{f^{-1}(B)}(x) = V_B(f(x)) \quad \forall x \in X$

Definition 2.6:[5] Let f be a mapping from a set into a set Y. Let A be a vague set Y. Then the image of A i.e. f[A] is the vague set in Y given by

$$V_{f[A]}(y) = \begin{cases} \sup\{V_A(z) : z \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \ne \phi \\ [0,0], & \text{otherwise} \end{cases}$$

for all y ∈ Y, where $f^{-1}(y) = \{x / f(x) = y\}$.

3. Vague submodule

Definition 3.1: Let V_A = $[t_A, 1-f_A]$ and V_B = $[t_B, 1-f_B]$ be two vague submodules in M. Then we define their sum $V_A + V_B$ as the vague submodule $V_{A+B} = \{x, [t_{A+B}(x), 1 - f_{A+B}(x)] : x \in M\}$ where for each $x \in M$ $t_{A+B}(x) = \vee\{t_A(y) \wedge t_B(z) : y, z \in M, x = y + z\}$ *and* $1 - f_{A+B}(x) = \vee\{1 - f_A(y) \wedge 1 - f_B(z) : y, z \in M, x = y + z\}$

Definition 3.2: Let VA = [tA,1-fA] be a vague submodule in M. Then $-V_A = [t_{-A}, 1 - f_{-A}]$, a vague submodule in M, is defined as $-V_A = \{x, [t_{-A}(x), 1 - f_{-A}(x)] : x \in M\}$ where, $V_{-A}(x) = V_A(-x)$

Definition 3.3: For a vague submodule VA = $[t_A, 1-f_A]$ in M and for any r ∈ R, we define the vague submodule $rV_A = [t_{rA}, 1 - f_{rA}]$ in M as $rV_A = \{(x, [t_{rA}, 1 - f_{rA}]) : x \in M\}$ where for each x ∈ M $V_{rA}(x) = \vee \{V_A(y) : y \in M, x = ry\}$

Definition 3.4: A vague set VA in M is called a vague submodule of M if for every xy ∈M and r ∈R, the following conditions are satisfiebd

(i) $V_A(0) = 1$.
(ii) $V_A(x+y) \ge V_A(x) \wedge V_A(y)$, for all x,y ∈ M;
(iii) $V_A(rx) \ge V_A(x)$, for all r ∈ R and x ∈ M;

Theorem 3.5: If VA = [tA,1-fA] is a vague submodule in a R- module M, then $1 \cdot V_A = V_A$ and $(-1) \cdot V_A = -V_A$

Proof: It is clear that $1 \cdot V_A = V_A$ .Now $(-1) \cdot V_A(x) = [t_{(-1)A}(x), 1 - f_{(-1)A}(x)]$ $t_{(-1)A}(x) = \vee \{t_A(y) : y \in M, x = (-1)y\} = \vee \{t_A(y) : y \in M, y = -x\} = t_A(-x) = t_{-A}(x)$.
Similarly we get $1 - f_{(-1)A}(x) = 1 - f_{-A}(x) \quad \forall x \in M$. Hence $(-1) \cdot V_A = -V_A$.

Theorem 3.6: If $V_A = [t_A, 1-f_A]$ and $V_B = [t_B, 1-f_B]$ is a vague submodules in M with $A \subseteq B$ then $rA \subseteq rB$ for any $r \in R$.

Proof: Since $A \subseteq B$, we have $V_A(x) \le V_B(x) \ \forall x \in M$. Also for any $r \in R$ rVA = =[tr_A,1-fr_A] and rV_B= [tr_B,1-fr_B]. Now $t_{rA}(x) = \vee \{t_A(y) : y \in M, x = ry\}$ $\le \vee \{t_B(y) : y \in M, x = ry\} = t_{rB}(x) \ \forall x \in M$ Similarly we can obtain for $1 - f_{rA}(x) \le 1 - f_{rB}(x) \ \forall x \in M$. Hence $rA \subseteq rB$

Theorem 3.7: If V_A = [t_A,1-f_A] is a vague submodule in M. Then $r(sV_A) = (rs)V_A$ for any $r \in R$

Proof: We have $r(sV_A) = [t_{r(sA)}, 1 - f_{r(sA)}]$ and $(rs)V_A = [t_{(rs)A}, 1 - f_{(rs)A}]$. Now,

$$t_{r(sA)}(x) = \vee\{t_{sA}(y) : y \in M, x = ry\} = \vee\{\vee\{t_A(z) : z \in M, y \in M, x = ry\}$$
$$= \vee\{t_A(z) : z \in M, x = r(sz)\} = \vee\{t_A(z) : z \in M, x = (rs)z\} = t_{(rs)A}(x), \forall x \in M.$$

Similarly we can show that $1 - f_{r(sA)}(x) = 1 - f_{(rs)A}(x) \ \forall x \in M$. Hence $r(sV_A) = (rs)V_A$ =for any $r \in R$

Theorem 3.8: If V_A = [t_A,1-f_A] and V_B = [t_B,1-f_B] are vague submodules in M, then $r(V_A + V_B) = rV_A + rV_B$ for any $r \in R$

$\subseteq B$

Proof: we have $(r(V_A + V_B)) = [t_{r(A+B)}, 1 - f_{r(A+B)}]$ and $r(V_A) + r(V_B) = [t_{rA+rB}, 1 - f_{rA+rB}]$
Now,

$$t_{r(A+B)}(x) = \vee\{t_{A+B}(y) : y \in M, x = ry\}$$
$$= \vee\{\vee\{t_A(z_1) \wedge t_B(z_2) : z_1, z_2 \in M, y = z_1 + z_2\} : y \in M, x = ry\}$$
$$= \vee\{t_A(z_1) \wedge t_B(z_2) : z_1, z_2 \in M, x = rz_1 + rz_2\}$$
$$= \vee\{(\vee\{t_A(z_1) : z_1 \in M, x_1 = rz_1\}) \wedge (\vee\{t_B(z_2) : z_2 \in M, x_2 = rz_2\}) : x_1 + x_2 = x\}$$
$$= \vee\{t_{rA}(x_1) \wedge t_{rB}(x_2) : x_1, x_2 \in M, x_1 + x_2 = x\}$$
$$= t_{rA+rB}(x), \ \forall x \in M.$$

Similarly we can show that $1 - f_{r(A+B)}(x) = 1 - f_{rA+rB}(x), \ \forall x \in M$ Hence $r(V_A + V_B) = rV_A + rV_B$

Theorem 3.9: If V_A = [t_A,1-f_A] and V_B = [t_B,1-f_B] are vague submodules in M, then $t_{rA+sB}(rx + sy) \ge t_A(x) \wedge t_B(y)$ *and* $1 - f_{rA+sB}(rx + sy) \ge 1 - f_A(x) \wedge 1 - f_B(y)$

Proof: We have $t_{rA+sB}(rx + sy) = \vee\{t_{rA}(z_1) \wedge t_{sB}(z_2) : z_1, z_2 \in M, z_1 + z_2 = rx + sy\}$
$$\ge t_{rA}(rx) \wedge t_{sB}(sy) \ge t_A(x) \wedge t_B(y) \ \forall x, y \in M; r, s \in R.$$

Similarly, $1 - f_{rA+sB}(rx + sy) \ge 1 - f_A(x) \wedge 1 - f_B(y) \ \forall r, s \in R.$

Theorem 3.10: If V_A = [t_A,1-f_A], V_B = [t_B,1-f_B] and V_C = [t_C,1-f_C] are vague submodules in M, then for any $r, s \in R$.

(i) $t_C(rx + sy) \ge t_A(x) \wedge t_B(y) \ \forall x, y \in M \Leftrightarrow t_{rA+sB} \subseteq t_C$.
(ii) $1 - f_C(rx + sy) \ge 1 - f_A(x) \wedge 1 - f_B(y) \ \forall x, y \in M \Leftrightarrow 1 - f_{rA+sB} \subseteq 1 - f_C$.

Proof: Suppose we have $t_C(rx + sy) \ge t_A(x) \wedge t_B(y) \ \forall x, y \in M$ Then

$$t_{rA+sB}(z) = \vee\{t_{rA}(z_1) \wedge t_{sB}(z_2) : z_1, z_2 \in M, z_1 + z_2 = z\}$$
$$= \vee\{(\vee\{t_A(x), x \in M, z_1 = rx\}) \wedge (\vee\{t_B(y) : y \in M, z_2 = sy\}) : z_1, z_2 \in M, z_1 + z_2 = z\}$$
$$= \vee\{t_A(x) \wedge t_B(y), x, y \in M, rx + sy = z\}$$
$$\le \vee\{t_C(rx + sy) : x, y \in M, rx + sy = z\} \le t_C(z), \ \forall z \in M.$$

Thus $t_{rA+sB} \subseteq t_C$.ow conversely suppose $t_{rA+sB} \subseteq t_C$. Then $t_C(rx + sy) \ge t_{rA+sB}(rx + sy) \ge t_A(x) \wedge t_B(y)$. by the above theorem. Similarly we can show that $1 - f_C(rx + sy) \ge 1 - f_A(x) \wedge 1 - f_B(y) \ \forall x, y \in M \Leftrightarrow 1 - f_{rA+sB} \subseteq 1 - f_C$.

Theorem 3.11: If V_Ai = [t_Ai,1-f_Ai] , i∈J, is a collection of vague submodules in an R- module M, then $r(\bigcup_{i \in J} V_{A_i}) = \bigcup_{i \in J}(rV_{A_i})$ for any $r \in R$.

Proof: We have $r(\bigcup_{i \in J} V_{A_i}) = [t_{r(\bigcup_{i \in J} A_i)}, 1 - f_{r(\bigcup_{i \in J} A_i)}]$ and $\bigcup_{i \in J}(rV_{A_i}) = [t_{\bigcup_{i \in J}(rA_i)}, 1 - f_{\bigcup_{i \in J}(rA_i)}]$. Now

$$t_{r(\bigcup_{i \in J} A_i)}(x) = \vee\{t_{\bigcup_{i \in j} A_i}(y) : y \in M, x = ry\} = \vee\{\bigcup_{i \in j} t_{A_i}(y) : y \in M, x = ry\} = \bigcup_{i \in j} t_{rA_i}(x)$$
$$= t_{(\bigcup_{i \in J} rA_i)}(x) \ \forall x \in M.$$

Hence $t_{r(\bigcup_{i \in J} A_i)} = t_{\bigcup_{i \in j}(rA_i)}$ .Similarly $1 - f_{r(\bigcup_{i \in J} A_i)} = 1 - f_{\bigcup_{i \in j}(rA_i)}$. This complets the proof.

Theorem 3.12: Let M and N be two R- modules and f be a homomorphism of M into N. Let r,s∈R and V_A = [t_A,1-f_A], V_B = [t_B,1-f_B] be two vague set in M. Then,

(i) $f(V_A + V_B) = f(V_A) + f(V_B)$
(ii) $f(rV_A) = rf(V_A)$
(iii) $f(rV_A + sV_B) = rf(V_A) + sf(V_B)$

Proof: (i) we have $f(V_A + V_B) = \{(y, [t_{f(A+B)}(y), 1 - f_{f(A+B)}(y)]) : y \in N\}$ and $f(V_A) + f(V_B) = \{(y, t_{f(A)+f(B)}(y), 1 - f_{f(A)+f(B)}(y) : y \in N\}$

Let y∈N. If $f^{-1}(y) = \phi$ then $t_{f(A+B)} = 0$, Also,

$$t_{f(A+B)}(y) = \vee\{t_{f(A)}(y_1) \wedge t_{f(B)}(y_2) : y_1, y_2 \in N, y = y_1 + y_2\}$$
$$= 0, \quad \text{since } f^{-1}(y_1) = \phi \text{ or } f^{-1}(y_2) = \phi \text{ as } f^{-1}(y) = \phi.$$

Thus, $t_{f(A+B)}(y) = t_{f(A)+f(B)}(y)$ *if* $f^{-1}(y) = \phi$.if $f^{-1}(y) \ne \phi$ then also we have,

$$t_{f(A+B)}(y) = \vee\{t_{A+B}(x) : x \in M, y = f(x)\}$$
$$= \vee\{\vee\{t_A(x_1) \wedge t_B(x_2) : x_1, x_2 \in M, x = x_1 + x_2\} : x \in M, y = f(x)\}$$
$$= \vee\{\vee\{t_A(x_1) \wedge t_B(x_2) : x_1, x_2 \in M, z_1 = f(x_1), z_2 = f(x_2)\} : z_1, z_2 \in N, y = z_1 + z_2\}$$
$$= \vee\{(\vee\{t_A(x_1) : x_1 \in M, z_1 = f(x_1)\}) \wedge (\vee t_B(x_2) : x_2 \in M, z_2 = f(x_2)\}) : z_1, z_2 \in N, y = z_1 + z_2\}$$
$$= \vee\{t_{f(A)}(z_1) \wedge t_{f(B)}(z_2) : z_1, z_2 \in N, y = z_1 + z_2\}$$
$$= t_{f(A)+f(B)}(y)$$

Thus in any case we get $t_{f(A+B)}(y) = t_{f(A)+f(B)}(y)$. Similarly we can obtain $1 - f_{f(A+B)}(y) = 1 - f_{f(A)+f(B)}(y) \ \forall y \in N$ Hence we have $f(V_A + V_B) = f(V_A) + f(V_B)$

(ii) Now,

$f(rV_A) = \{(y, [t_{f(rA)}(y), 1 - f_{f(rA)}(y)]) : y \in N\}$ *and* $rf(A) = \{(y, [t_{rf(A)}(y), 1 - f_{rf(A)}(y)) : y \in N\}$
Let y∈N. Then if $f^{-1}(y) = \phi$ then obviously $t_{f(rA)}(y) = 0$. Also,
$t_{rf(A)}(y) = \vee\{t_{f(A)}(v) : v \in N, y = rv\} = 0$, since $f^{-1}(v) = \phi$ as $f^{-1}(rv) = f^{-1}(y) = \phi$. Thus
$t_{f(rA)}(y) = t_{rf(A)}(y)$ if $f^{-1}(y) = \phi$. Now if $f^{-1}(y) \ne \phi$, then

$$t_{f(rA)}(y) = \vee\{t_{rA}(x) : x \in M, y = f(x)\} = \vee\{\vee\{t_A(u) : u \in M, x = ru\} : x \in M, y = f(x)\}$$
$$= \vee\{\vee\{t_A(u) : u \in M, v = f(u)\} : v \in N, y = rv\}$$
$$= \vee\{t_{f(A)}(v) : v \in N, y = rv\} = t_{rf(A)}(y)$$

Thus we get $t_{f(rA)}(y) = t_{rf(A)}(y) \ \forall y \in N$ Similarly $1 - f_{f(rA)}(y) = 1 - f_{rf(A)}(y) \ \forall y \in N$ Hence we get $f(rV_A) = rf(V_A)$

(iii) This follows from (i) and (ii).

Theorem 3.13: Let M and N be two R-modules and f be a homomorphism of M into N. If $V_A = [t_A, 1 - f_A]$ is a vague submodule of M, then *f(A)* is a vague submodule of N.

Proof: We have $f(V_A) = \{(y, [t_{f(A)}(y), 1 - f_{f(A)}(y)]) : y \in N\}$ First of all we note that $t_{f(A)}(0) = \vee\{t_A(x) : x \in M, f(x) = 0\} = t_A(0) = 1$ To prove the second condition of a vague submodule , Let $y_1, y_2 \in N$. *If* $f^{-1}(y_1) = \phi$ *or* $f^{-1}(y_2) = \phi$ then correspondingly $t_{f(A)}(y_1) = 0$ *or* $t_{f(A)}(y_2) = 0$. So in this case $t_{f(A)}(y_1) \wedge t_{f(A)}(y_2) = 0$ and so $t_{f(A)}(y_1 + y_2) \ge t_{f(A)}(y_1) \wedge t_{f(A)}(y_2)$ Now if $f^{-1}(y_1) \ne \phi \ne f^{-1}(y_2)$, then

$$t_{f(A)}(y_1 + y_2) = \vee\{t_A(x) : x \in M, y_1 + y_2 = f(x)\}$$
$$= \vee\{t_A(x_1 + x_2) : x_1, x_2 \in M, y_1 + y_2 = f(x_1 + x_2)\}$$
$$\ge \vee\{t_A(x_1 + x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$$
$$= \vee\{t_A(x_1 + x_2) : x_1, x_2 \in M, y_1 = f(x_1), y_2 = f(x_2)\}$$
$$\ge \vee\{t_A(x_1) \wedge t_A(x_2) : x_1, x_2 \in M, y_1 = f(x_1), y_2 = f(x_2)\}$$
$$\ge (\vee\{t_A(x_1) : x_1 \in M, y_1 = f(x_1)\}) \wedge (\vee\{t_A(x_2) : x_2 \in M, y_2 = f(x_2)\}$$
$$= t_{f(A)}(y_1) \wedge t_{f(A)}(y_2)$$

Thus $t_{f(A)}(y_1 + y_2) \ge t_{f(A)}(y_1) \wedge t_{f(A)}(y_2) \ \forall \ y_1, y_2 \in N$. Similarly we can prove that $1 - f_{f(A)}(y_1 + y_2) \ge 1 - f_{f(A)}(y_1) \wedge 1 - f_{f(A)}(y_2) \ \forall \ y_1, y_2 \in N$

Now to prove the third condition of a vague submodule, let $y \in N$. If $f^{-1}(y) = \phi$ then
$t_{f(A)}(y) = 0$. So in this case $t_{f(A)}(ry) \ge t_{f(A)}(y)$. Now if $f^{-1}(y) \ne \phi$, then

$$t_{f(A)}(ry) = \vee\{t_A(x) : x \in M, ry = f(x)\} \ge \vee\{t_A(rx_1) : rx_1 \in M, ry = f(rx_1)\}$$
$$\ge \vee\{t_A(rx_1) : x_1 \in M, x_1 = f^{-1}(y)\} = \vee\{t_A(rx_1) : x_1 \in M, y = f(x_1)\}$$
$$\ge \vee\{t_A(x_1) : x_1 \in M, y = f(x_1)\} = t_{f(A)}(y)$$

Thus $f(V_A)$ is a vague submodule of N.

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