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A note on n-semiregular graphs

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ABSTRACT A graph in which every vertex has equal number of vertices at distance 2 is called semiregular graph. In particular, if every vertex has exactly n number of vertices at distance 2 then it is called n-semiregular graph. Every regular graph need not be semiregular and vice-versa, but every complete graph is 0-semiregular. In this paper we have constructed semiregular graphs from complete graphs and also we have discussed the n-semiregularity of the well-known graphs (classical graphs) such as circulant graphs and vertex transitive graphs.

KEYWORDS : semiregular graphs; Kn + 1 graph ; product graph

1. Introduction

Just like the regular graph in which each vertex is at distance 1 away from the same number of vertices, the graph in which each vertex is at distance 2 away from the same number of vertices is called semiregular graph. Some authors also defined the graphs in which the degree of each vertex is either r or r+1 are semiregular graphs [5]. In this paper, by n-semiregular graph we mean the graph in which each vertex is at distance 2 away from n number of vertices in that graph. Originally, the concept of semiregular graphs evolved from the combination graphs studied by Balaban [2] in the year 1972 and convolution graphs by Kerek [6] in the year 1974. In the year 2002, Alison Northup [1] defined the n-semiregular graphs and discussed an algorithm to construct an n-semiregular graph for a given integer n. The study of these graphs have further got momentum after the publication of papers "The distance degree regular graphs" by Bloom G.S etal [3] and "How to define an irregular graph" by Chartrand etal [4].

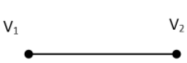
In this paper, the n-semiregularity of some classical graphs have been discussed in detail.

2. n-Semiregular Graphs

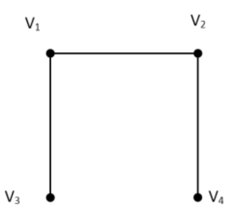
A simple graph G is said to be n-semiregular graph if each vertex in G has exactly n vertices at distance 2 in G.. It is also called (2,n) – regular graph.

2.1 *Examples*

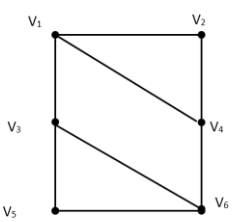
The graphs given in fig. 1 are examples of some n- semiregular graphs



0-semiregular

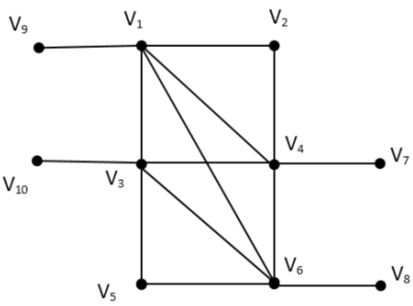


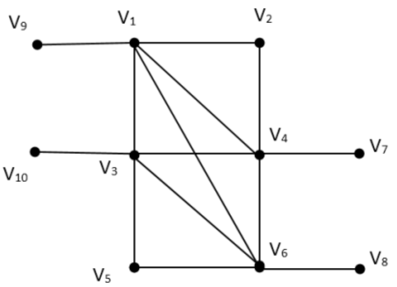
1-semiregular



2-semiregular

3-semiregular





4-semiregular

Fig. 1 Some n-semiregular graphs

All n-semiregular graphs are not regular. The graphs which are regular are denoted as (r,2,n) – regular graphs i.e., each vertex in the graph is at a distance 1 away from exactly r vertices and at a distance 2 from exactly n vertices. C. Sekar and N.R. Shanthimaheswari [7] have studied some properties of (r, 2, n)-regular graphs. An n-semiregular which is not regular is simply denoted as (2,n) regular graph

2.2 *Examples*

The graph given in fig.2 a is (3,2,3)-regular graph and fig.2 b is (2,3)-regular graph

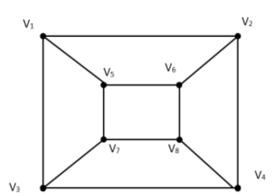


Fig 2 . a (3,2,3) – regular graph

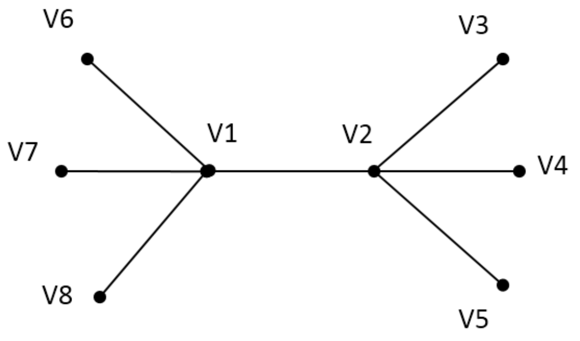


Fig 2 . b (2,3) – regular graph

The following theorem gives the necessary condition for a graph to be a n-semiregular.

2.3 Theorem

Let G be a semiregular graph, and let u and v be any two vertices of deg m. If there is a vertex x of deg n adjacent to u then there is a vertex y of deg n, adjacent to the vertex v.

Proof

Let G be a semiregular graph. Let deg u = deg v = m. Also let x and y are vertices adjacent to u and v respectively such that deg x = k and deg y = l where k ≠ l. For simplicity, first let us assume that deg u = deg v = 1. Then if deg x ≠ deg y, then the number of vertices which are at distance 2 for u and v will not be the same. It is a contradiction to the assumption that G is semiregular. Hence the theorem holds.

Next let us assume that deg u = deg v > 1, then the number of vertices which are at distance 2 to u through the vertex x is k. Similarly the number of vertices which are at distance two to v through y is l. Since k ≠ l, the number of vertices which are at distance two from u and v are not same. Again this is a contradiction. Hence the theorem.

The converse of the theorem 2.4 is not true; for example, consider the Grotsch's graph given in fig.3

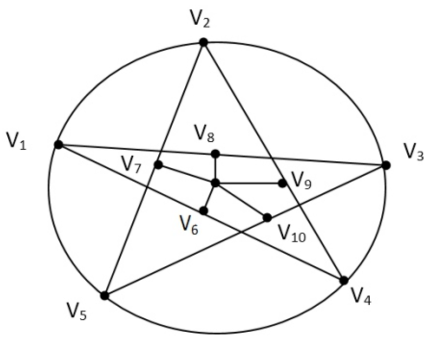


Fig. 3

There are vertices of deg 3, deg 4 and deg 5. Also the number of vertices of deg 3, deg 4 and deg 5 are 5, 5, 1 respectively.

Let αij denote the number adjacent vertices of deg j to the vertex of deg i. Then it can be found that α11 = 0 ; $\alpha_{12} = 2$; $\alpha_{13} = 1$; $\alpha_{21} = 2$; $\alpha_{22} = 2$; $\alpha_{23} = 0$; $\alpha_{31} = 5$; $\alpha_{32} = 0$; $\alpha_{33} = 0$.

But the above graph is not semiregular.

2.4. Theorem

If a graph G is semiregular then given any vertex u of deg m, the sum of degrees of adjacent vertices is a constant independent of the choice of u.

Proof

Let G be a semiregular graph. Let u and v are any two vertices in G such that deg(u) = deg(v). From the above theorem, both the vertices u and v will have adjacent vertices of same degrees. Hence the sum of the degrees of adjacent vertices of u and v is a constant. Hence the theorem

2.5 Example

As an illustration to theorem 2.4, consider the 4-semiregular graph given in fig.1. The following table gives the sum of degrees of adjacent vertices of a given vertex of the 4-semiregular graph

Vertices of deg 1	Adjacent vertices	Sum of degrees of adjacent vertices
V_7	V_4	5
V_8	V_6	5
V_9	V_1	5
V_{10}	V_3	5

Vertices of deg 2	Adjacent vertices	Sum of degrees of adjacent vertices
V_2	V_1 , V_4	10
V_5	V_3 , V_6	10

Vertices of deg 5	Adjacent vertices	Sum of degrees of adjacent vertices
V_1	V_9 , V_2 , V_4 , V_6 , V_3	18
V_3	V_{10} , V_5 , V_6 , V_4 , V_1	18
V_4	V_7 , V_2 , V_1 , V_3 , V_6	18
V_6	V_8 , V_5 , V_3 , V_1 , V_4	18

2.6. Theorem

A simple connected graph is 0-semiregular if and only if it is a complete graph.

Proof

Let G be a connected 0-semiregular. Then for all vertex u in G, there is no vertex v in G such that d(u, v) = 2. Hence there is no vertex w in G such that d(u, w) > 2, otherwise, it would lead to a vertex at distance 2 from u. Hence for all vertices u, v in G, d(u, v) < 2, i.e., d (u, v) is either 0 or 1. d(u,v) = 0 is also not possible, if so, the vertices u and v disconnects the graph. This contradicts the fact that G is connected. Thus, for all vertices u and v in G, d(u, v) = 1. Hence G is a complete graph.

Conversely, let G is complete. Then for all vertices u, v in G, d (u, v) = 1. Hence G is 0-semiregular.

From the following section it can be seen that an n-semiregular graph for any positive integer n can be constructed from a complete graph.

3. Construction of n-semiregular graphs

3.1 Definition

The graph constructed from K2 by adding n pendent vertices each at both the vertices of K2 is called n-Barbell graph. It is denoted as k2 + n.

3.2 Theorem

The n-Barbell graph is n-semiregular

Proof

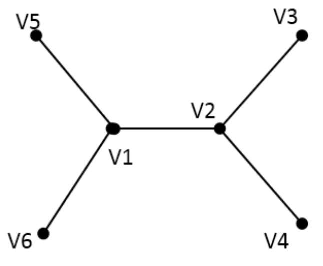
Let $V(K_2) = \{u,v\}$; Also let $K_2 + n$ is the graph obtained by adding the pendent vertices $w_1, w_2, \dots wn$ at u, and $wn+1, \dots w_2n$ at v. The following table gives the vertices which are at distance 2 from the given vertex

Vertex	Vertices which are at distance 2
u	$wn+_1, \dots w_2n$
v	$w1, w_2, \dots wn$
wi, $i=_{1,2,}\dots n$	$v, w_1, w_2, \dots wi-_1, wi+_1, \dots wn$
$wn+i$, $i=_{1,2,}\dots n$	$u, wn+_1, \dots wn+i-_1, wn+i+_1, \dots w_2n$

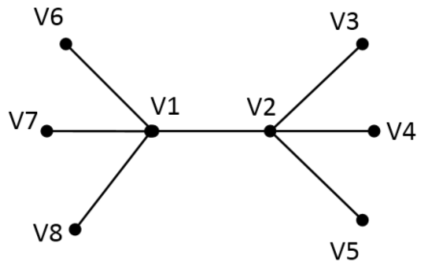
Hence every vertex has exactly n vertices at distance 2. Therefore K2 + n is an n-semiregular graph.

3.3 Examples

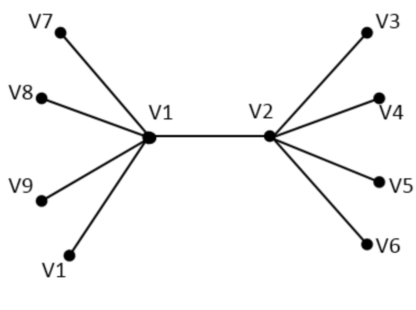
It can be easily seen that the 2, 3,4-Barbell graphs given in fig.4 are respectively 2,3,4-semiregular graphs.



2-semiregular graph



3-semiregular graph



4-semiregular graph

Fig.4 Barbell graphs

3.4 Definition

Let Kn be a complete graph. The graph obtained from Kn by adding pendent vertices at each vertex of Kn is called as Kn + 1 graph.

3.5 Theorem

graph Kn + 1 is (n-1)-semiregular.

Proof

Consider a complete graph Kn, with vertices $u_1, u_2, \dots$ un. Add the pendent vertices $v_1, v_2, \dots$ vn respectively at $u_1, u_2, \dots$ un. It can be seen that for every vertex u_i, the (n-1) vertices $v_1, v_2, \dots v(i-1), v(i+1), \dots$ vn are at distance 2. Similarly for every vertex v_i, the (n-1) vertices $u_1, u_2, \dots u(i-1), u(i+1), \dots$ un are at distance 2. i.e., each vertex in Kn + 1 has exactly (n-1) vertices at distance 2.

Hence Kn + 1 is (n-1)-semiregular.

3.6 Examples

The graphs given in Fig.5 and fig.6 are 2-semiregular and 3-semiregular graphs constructed respectively from the complete graphs K_3 and K_4.

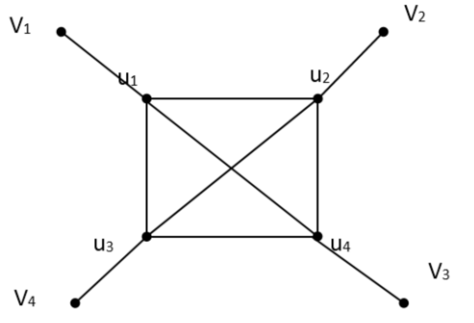
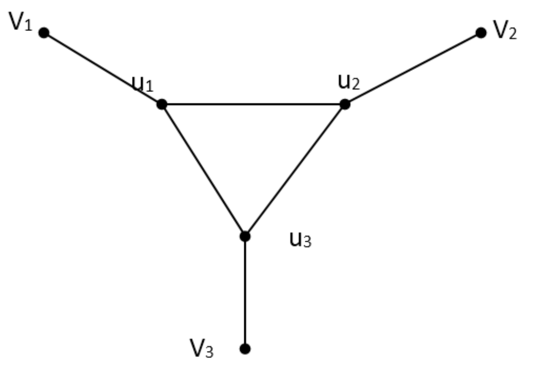


Fig. 5 3-semiregular graph $K_4 + 1$



3.8 Theorem

For any bijection map $f : V(K_n') \to V(K_n'')$, the grap $K_n' \bullet_f K_n''$ is (n-1)-semiregular.

Proof

Let V(Kn)={$u_1, u_2, \dots$ un} and V(Kn)={$v_1, v_2, \dots$ vn} . Let f : V(Kn) V(Kn is bijective . Note that no vertex ui is at distance 2 to any vertex in Kn similarly no vertex vi is at distance 2 to any vertex in Kn . Also for every vertex ui in Kn there is exactly one vertex vi in Kn is adjacent to ui, by means of an edge added with respect to the bijective map f. i.e., for each vertex ui except f(ui), all other vertices in Kn" are at distance 2. The same argument is also applicable to all other vertices in Kn, and also all the vertices in Kn. Hence Kn •f Kn" is (n-1)-semiregular.

3.9 Example

The graphs given in fig.7 and fig.8 are K_3 f K_3 and K_4f K4" graphs obtained with respect to the bijection f(ui)= vi respectively.

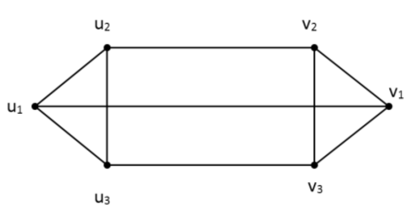


Fig. 7. $K_3 f$ K_3" Graph

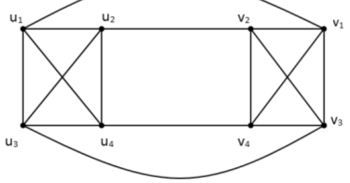


Fig.8 K_4'•f K_4" Graph

4. Circulant Graphs

4.1 Definition

Let r_1, r_2,…rk be the reduced residue system modulo m. A graph G with vertex set {v_0, v_1, v_2,…vm-1} in which vi is adjacent to vj if and only if i - j (mod m) = r.n , for some n satisfying 1 ≤ n ≤ k, is called circulant graph corresponding to the integer m. It is denoted as cir (m)

4.2 Observations

i. We know that the integer 1 is relatively prime to any integer m. Hence vi , vi+1 are always adjacent vertices in the circulant graph corresponding to the integer m.

ii. Also 0 and m-1 are always relatively prime implies that v0 and vm-1 are always adjacent vertices. Hence v0, v1, . . . vm-1 form a cycle in the circulant graph cir(m).

iii. In a circulant graph any two vertices are either adjacent or at distance 2

4.3 Theorem

The circulant graph cir(p), p is a prime is 0-semiregular.

Proof

Let p be a prime. The integers relatively prime to p are 1,2,3,…p-1. Let the vertex set be {v0, v1,v2,…vp-1}. Then for any two integers i,j satisfying 0 ≤ i , j ≤ p-1, i-j(mod p) = r ∈{1,2,…p-1}. Hence every v¬i is adjacent to vj . i.e., the circulant graph cir(p) is a complete graph and hence it is 0-semiregular.

4.4 Examples

The circulant graphs cir(7), cir(8), cir(9) are given in fig. 9, fig. 10 and fig. 11

4.3 Theorem

The circulant graph cir(p), p is a prime is 0-semiregular.

Proof

Let p be a prime. The integers relatively prime to p are 1,2,3,…p-1. Let the vertex set be {v0, v1,v2,…vp-1}. Then for any two integers i,j satisfying 0 ≤ i , j ≤ p-1, i-j(mod p) = r ∈{1,2,…p-1}. Hence every v¬i is adjacent to vj .

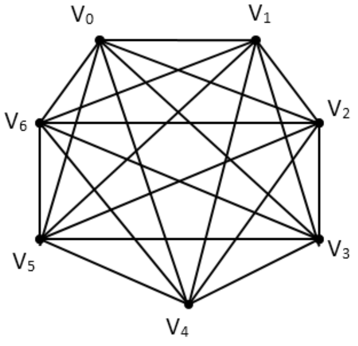


Fig.9 cir(7)

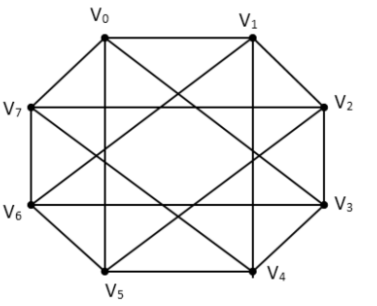


Fig. 10 cir(8)

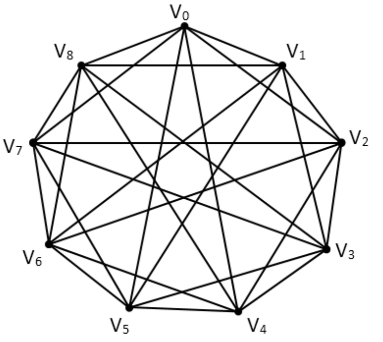


Fig.11 cir(9)

On generalization of the examples given above, we have the following theorems.

4.5 Theorem

Let m be an odd integer, then the circulant graph cir(m) is -semiregular, where is the number of integers not relatively prime to m.

Proof

Let cir(m) be the circulant graph with v0, v1, . . . vm-1 as vertices. The theorem is trivially true for a prime odd integer, because the number of integers not relatively prime to a prime is zero i.e., = 0. Hence the cir(m) is 0-semiregular when m is prime. Therefore let m be a non-prime odd integer and r1 , r_2 , . . . rk are prime factors of m. Then the number of integers not relatively prime to m is =∑_(i=1)(m/ri - 1). Hence the number of integers relatively prime to m is (m--1), and let them be k1, k2, . . . km-α-1. As the difference between each of the integers k1, k2, . . . km-α-1 and 0 is relatively prime to m, then by definition of circulant graph v0 is adjacent to v(k1,) v(k2,) . . . v(k(m--1)). Let s_1, s_2, . . . sα be the integers not relatively prime to m. Then v(s1), v(s2), . . . vs) are not adjacent to v_0. Hence v(s_1), v(s_2), . . . v(s) are at distance 2 from v_0. i.e., there are α number of vertices at distance 2 from v_0.

It can be easily seen that the difference between k_1+1, k_2+1, . . . km-α-1+1 and 1 is the same as the difference between k_1, k_2, . . . km-α-1 and 0. i.e., the difference is relatively prime to m. Here the summation is over mod m.

Hence v_1 is adjacent to the same number of vertices as v_0 and also v_1 has the same number of vertices at distance 2 as v_0. By proceeding in this manner, it can be found that, every vertex in cir(m) has same number of adjacent vertices and also same number of vertices at distance 2 as v_0. Hence cir(m) is -semiregular when m is an odd integer.

4.6 Theorem

Let m be an even integer, then the circulant graph cir(m) is ((m-2)/2)-semiregular.

Proof

It is clear that vi , vi+1 are always adjacent in a circulant graph cir(m), for any integer m. Suppose, if m is an even integer then vi and vj are at distance 2 to each other ,when i – j is even integer. Hence if v0, v1, . . . vm-1 are the vertices in cir(m), then v0, v2, . . . vm-2 are at distance 2 to each other and v1, v3, . . . vm-1 are also at distance 2 to each other. i.e., each vertex has exactly ((m-2)/2) vertices at distance 2 in cir(m), when m is even. Hence cir(m), when m is even is ((m-2)/2)-semiregular. Hence the theorem.

5. Vertex Transitive Graphs

5.1 Definition

A graph G is vertex-transitive if for every pairs of vertices vi and vj in G there is an automorphism on G mapping vi to vj .

It can be easily seen that every one-one and onto mapping defined on the set of vertices of a complete graph to itself is an automorphism. Also, such an automorphism exists mapping any two vertices, being always adjacent to each other. Hence every complete graph is vertex transitive.

5.2 Theorem

Every cycle is vertex transitive

Proof

Let Cn be the cycle with vertex set V= {v1,v2,. . . vn}. Consider the vertices vi and vj and let d(vi,vj) ≤ ⌊n/2⌋. Let m is an integer such that d(v_i,v_j)=m≤⌊n/2⌋. Let f be a mapping defined on V to itself such that f(u) = v if d(u,v) = m. In Cn, for every vertex u, there are always two vertices v1 and v2 such that d(u,v1) = d(u,v2) = m. Hence choose any one arbitrarly, so that f is a one-one and onto mapping. Note that such a mapping always exists in Cn, for all n ≥3

5.3 Theorem

A connected vertex-transitive graph is regular

Proof

Let G be connected vertex-transitive graph, and vi ,vj ∈ V(G).Let f be an automorphism mapping vi to vj .If possible assume that |N(v(i))|=m and |N(v(j))|=n and m≠n .Let e_1 , e_2 , … em are edges incident to vi and f_1 , f_2 , … fn are edges incident to vj . Also let m > n and ek'=(v_i,u_k) and ek''=(vj,wl)

where k = 1,2, . . . m ; l = 1,2, . . . n. Then for every (vi , uk) , (f(vi) , f(uk)) = (vj , wl¬) for some l. Since m > n , all (vj , wl) cannot be different for all uk's .Hence there exist integers k_1 , k_2 , . . . kt , where 1≤t≤m such that (f(vi) , f(uk_1)) = (f(vi) , f(uk_2)) = (f(vi) , f(uk_3)) = . . . = (vj , wl) for some finite integer l. i.e., f(uk_1) = f(uk_2) = . . . = f(ukt) = wl. This contradicts the fact that f is one–one function. Next, let us assume that m < n. Thus there exist an integer k such that (f(vi) , f(uk)) = (vj , wl1) = (vj , wl2)) = . . . = (vj , wls) i.e., f(uk) = wl1 = wl2 = . . . = wls.This is also a contradiction. Hence m = n is the only possibility. Therefore G is regular.

The converse of the above theorem is not true. In the literature, it is given that the Gray graph, an undirected cubic bipartite graph with 54 vertices and 81 edges is of 3-regular but not vertex-transitive.

5.4 Theorem

Every connected vertex-transitive graph is n-semiregular

Proof

Let G be a connected vertex–transitive graph. Let v_1 and v_2 are any two vertices in G. Also let there are n vertices u_1, u_2, …un in G such that d(v_1,ui) = 2.Then there are 'n' number of (v_1,ui)-paths each of length 2 in G. Let f be an automorphism on G mapping v_1 to v_2. There are n number of (v_2,f(ui))-paths, and of length 2. Hence, there are n vertices w_1,… wn such that f(ui) = wi , and all w's are exactly two distance away from v2. i.e., there are exactly n vertices at distance 2 from v2. Hence G is n-semiregular.

Combining theorem 5.2 and 5.4 and also from the earlier discussion, we have the following theorem.

5.5 Theorem

Every cycle is 2-semiregular, n ≥ 5.

6. Conclusion

In this attempt, some properties of semiregular graphs, and the semiregularity of certain classical graphs such as circulant graphs and vertex transitive graphs have been discussed. This attempt can also be extended to other well known graphs.

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