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COHOMOLOGICAL ASPECTS OF PRODUCT OF SPHERES

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ABSTRACT In this article I shall review the product of spheres in transformation groups to study the cohomological aspects of products of spheres. Initially we use some basic notations of assumptions then discuss about this product of two spheres and several n-spheres and shall also obtain some new results in this article.

KEYWORDS : Product of spheres, product of two spheres.

Introduction:

Throughout our discussion of transformation groups via cohomology theory, we adopt the following notations and assumptions.

p stands for either prime or zero. G is the cyclic group of order p where $p \neq 0$. For $p = 0$, G is the circle group. K is the prime field whose characteristic is p. Some times K is the coefficient domain for cohomology. X is a G-space that is to say X is a space on which G acts. F stands for fixed point of G on X. We use Sheaf theoretic cohomology in the cohomological theory [5].

Generally when we speak of spaces X either it is paracompact whose $\dim_K X < \infty$ or X is compact. We define $\dim_K X$ in the following fashion: If X is locally paracompact then we define $\dim_L X = \dim_{\phi,L} X$, where ϕ is paracompactifying and $E(\phi) = X$. Also recall that if ϕ and ψ are paracompactifiying and $E(\phi) \subset E(\psi)$ then any ψ-soft sheaf is ϕ-soft sheaf and $\dim_{\phi,L} X \leq \dim_{\psi,L} X$.

Thus softness and dimension with respect to a paracompactifying family depends only on the extent of the family.

We say that a space is locally paracompact if it is Hausdorff and point has a closed paracompact neighbourhood. Such a space has a paracompactifying family ϕ with $E(\phi) = X$. That is to say the set of all closed K such that K has a closed paracompact neighbourhood in X is such a family.

Since any type of fixed point set can occur if X is a Hilbert space therefore our space X assumes that X is paracompact and $\dim_K X < \infty$ or X is compact.

It is easy to observe that the space X is paracompact with $\dim_K X$ is finite is much easier to deal with than X is compact and it is enough for most applications. If X is infinite dimensional, certain results if X is compact have proven to be of value in the theory of compact semigruop. For the above reasons we shall deal both the cases.

We also assume that if X has certain type of cohomology then so does F. We use the following notations.

$X \sim Y$ (X is homeomorphic to Y) to mean $H^*(X,K) \approx H^*(Y,K)$ (isomorphic as a ring). The simplest type of space is an acyclic space that is a point. $X \sim$ point $\Rightarrow F \sim$ point.

The next simplest space is a sphere and $X \sim S^n \Rightarrow F \sim S^r$ for some $r \leq n$. Note that the coefficients used and the restrictions on the group acting are vital for these results.

The next simplest space, as far as the cohomology ring structure is concerned is a space whose ring is generated by one element and hence is a truncated polynonimal ring. These include the real, complex and quaternionic projective spaces $(P^n, \mathbb{C}P^n and\ \mathbb{Q}P^n)$ and the Cayley projective plane as well as other more exotic spaces. The first theorem regarding these is also due to Smith 14 who proved that $X = P^n \Rightarrow F \sim P^n + P^{n-r-1}$ or $F = \phi$ for the case $p = 2$, '+' denotes the disjoint union. The method used was to lift the action to a $Z_2 \oplus Z_2$ action on S^n and study the fixed points of elements of this group. J.C.Su used the same method to obtain the result for more general $X \sim P^n(p = 2)$ and also for $p = 0$ and $X \sim \mathbb{C}P^n$.

1. Preliminaries:

Lemma: 2.1

Let A be a locally constant sheaf of K-module on B_G. Then multiplication by $t: H^r(B_G,$ A $) \longrightarrow H^{r+d}(B_G,$ A$)$ ($d = \deg t$ = 1 for $p = 2$ and $d = \deg t$ = 2 otherwise) is an isomorphism for $r > 0$ and an epimorphism for $r = 0$. If A is constant then this is also an isomorphism for $r = 0$.

Proof:

If A is a constant sheaf then $H^*(B_G,$ A$) \approx H^*(B_G) \otimes$ A. For $p = 0$ any locally constant sheaf is constant since $\pi_1(B_G) = 0$.

Proposition:2.2

Suppose that $H^i(X) = 0$ for $i > n$. Then the product by $t \in H^d(B_G)$: $tU(.): H^r(X_G) \longrightarrow H^{r+d}(X_G)$ is an isomorphism for $r > n$ and an epimorphism for $r = n$. If G acts trivially on $H^*(X)$, then this is an isomorphism for $r \geq n$ ($r \geq n-1$ for $p = 0$).

Theorem: 2.3

If $H^i(X) = 0$ for $i > n$ and $H^i(F) = 0$ for $i > n$.

Theorem:2.4

If $H^i(X) = 0$ for $i > n$ then $H^i_\varphi\left((X - F)/G\right) = 0$ for $i > n$.

2. Product of spheres

We consider the case $X \sim S^n \times S^m$. This is the case in which fixed point sets can occur. This is the first case is which X need not be totally non homologous to zero in X_G when $F \neq \phi$. There are two possibilities one is G may act nontrivially on $H^n(X)$ if $n = m$, the another one is G may act trivially but the spectral sequence may be non trivial.

In this section we assume X to be totally non homologous to zero in X_G. The following cases are the only possibilities for F :

1. $F \sim S^q \times S^r$,
2. $H^*(F)$ generated by u, v in degree q with $u^2 \neq 0$, $v^2 \neq 0$, and
   $uv = 0$,
3. $F \sim p^3(q)$,
4., $F \sim pt + p^2(q)$
5. $F \sim S^q + S^r$ (q and / or r can be zero).

In this section our main result is the following.

Theorem: 3.1

Let $p = 2$ and $X \sim S^n \times S^m$ with $n \leq m$. Suppose $F \sim S^r + S^q$ with $n < r < q$. Then $q \leq m$. Moreover if $2^k < n$ or if $2^k = n$ and $S_a^{2k}: H^m(X) \longrightarrow H^{m+n}(X)$ is trivial.

Proof:

Let a and b be generators of $H^*(X)$ in degrees n and m respectively. Let x be a point in the S^r-component of F. Let $c \in H^0(F)$ be the generator for the S^q-component and let d and e be generators of $H^*(F)$ in degrees r and q respectively. Thus $cd = 0$ and $ce = e$. Let α and β in $H^*(X_G)$ represent a and b such that

$\eta_x(\alpha) = 0 = \eta_x(\beta)$. If $q > m$ then $j^*(\alpha)$ and $j^*(\beta)$ do not involve e. But then $j^*(\alpha\beta)$ also does not involve e and j^* could not be onto in high degrees. Thus $q \leq m$.

Now $j^*(\alpha)$ can not involve d or e, by degree, and since $j^*(\alpha) \neq 0$ and $\alpha \in Ker\,\eta_x$ we must have $j^*(\alpha) = t^n \otimes c$. The coefficient d and e in $j^*(\beta)$ must be non zero for otherwise j^* will not be onto in high degrees. Thus $j^*(\beta) = At^m \otimes c + t^{m-r} \otimes d + t^{m-n} \otimes c$ where $A = 0$ or 1.

By assumption we must have that $S_q^i(\beta)$ is dependent upon α and β over $H^*(B_G)$ for all $i \leq 2^k$.

But $$j^*(S_q^i\beta) = S_q^i j^*\beta = A\binom{m}{i}t^{m+i} \otimes c + \binom{m-r}{i}t^{m-r+i} \otimes d + \binom{m-q}{r}t^{m-q+1} \otimes c$$

$$\Rightarrow \binom{m-r}{i} = \binom{m-q}{i} (\operatorname{mod} 2) \text{ and } 1 \leq i \leq 2^s.$$

Let i run through powers 2 shows that $m - r = m - q(\operatorname{mod} 2^{k+1})$.

i.e., $2^{k+1} \mid (q - r)$.

It sees m likely that this theorem could be improved so that the conclusion $2^{k-1} \mid (q-r)$ is replaced by $2^{\phi(n)} \mid (q-r)$. However one can try this result.

Continuing the discussion of product of spheres.

We examine the case of $X \sim S^n \times S^n$ in which G acts non trivially on $H^*(X,K)$. For $X = S^n \times S^n$ only the cases $p = 2,3$ can occur. It is easy to see that Z + Z admits automorphisms of period k only for $k = 1,2,3,4,6$. Let A be an integral 2×2 matrix with $A^k = I$. Let λ, μ be the eigen values of A. Then $\lambda\mu = \det A = \pm 1$, $\lambda + \mu = trace\, A$ is an integer and λ, μ are k^{th} roots of 1. It follows that $\lambda\mu = 1$ (or $k = 2$) and that $\lambda + \mu = -2, -1, 0, 1$ or 2. These cases yield, respectively $k = 1,2,4,6$ and 1. The case $p = 3$, $X = S^n \times S^n$. Let i_1 be the map from $S^n \times (*) \longrightarrow S^n \times S^n$ and i_2 be the map from $(*) \times S^n \longrightarrow S^n \times S^n$ such that $S^n \times (*)$, $(*) \times S^n$ are contained in $S^n \times S^n$.

Let π_1, π_2 be the projections. Let $T : S^n \times S^n \longrightarrow S^n \times S^n$ be of period three and consider T_* on $H_n(S^n \times S^n, \text{Z})$. Let $\alpha_1, \alpha_2 \in H_n(S^n \times S^n)$ be the basis given by i_1 and i_2 respectively and suppose

$$\begin{Bmatrix} T^*(\alpha_1) = a\alpha_1 + b\alpha_2 \\ T^*(\alpha_2) = c\alpha_1 + d\alpha_2 \end{Bmatrix}$$

We have

$$\begin{Bmatrix} ad - bc = \det T_* = 1 \\ a + d = tr\, T_* = -1 \end{Bmatrix} \to (1)$$

Now consider the maps $\pi_i T : S^n \times S^n \longrightarrow S^n$ for $i = 1,2$. Computation shows that these have bidegress (a,c) and (b,d) respectively. The Hopf construction yields maps $S^{2n+1} \longrightarrow S^{n+1}$ with Hopf invariants ac and bd respectively. Thus for $n \neq 1,3,7$, ac and bd must be even (actually zero for n even). This is inconsistent with (1).

If $G = Z_3$ acts on $X = S^n \times S^n \ (n = 1,3,7)$ and is non trivial on $H^n(X, Z_3)$ then it can be show that $F \sim pt + S^r$ for some even $r < n$. In fact $\dim H^*(F) \leq 3$ and $\chi(F) \equiv 0 \pmod 3$ so that it need only be shown that F is nonempty and disconnected.

We shift our attention to the case $p = 2$. Here the involution T resembles cohomologically the interchange of factors of $S^n \times S^n$, in order that T^* be non trivial on $H^n(S^n \times S^n, Z_2)$. Thus one would expect F to resemble the diagonal S^n. To justify this we prove the following.

Theorem: 3.2

Let T be the involution on $X \sim S^n \times S^n$ such that $T^* \neq I$ on $H^n(X, Z_2)$. Then $F \sim S^n$ and the restriction $H^n(X) \longrightarrow H^n(F)$ is onto. Moreover if $X = S^n \times S^n$, $n \neq 1,3,7$ and if $\pi : X \longrightarrow S^n$ is the projection on either factor then $(\pi / F)^* : H^n(S^n) \longrightarrow H^n(F)$ is an isomorphism

Proof:

First it is clear that $F \sim S^r$, since $\chi(F) \equiv \chi(X) \equiv 0 \pmod 2$ and $\dim H^*(F) < 4$. Let $x \in H^n(X)$ with $y = T_* x \neq x$. Then x, y generate $H^n(X)$ and necessarily $x^2 = y^2 = 0$ (since coefficient are in Z_2). Also $x / F \neq 0$ so that $r = n$.

If $X = S^n \times S^n$ then let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of T^* on $H^n(S^n \times S^n, Z)$ with respect to the usual basis.

By the argument given above we find that ac and bd are even when $n \neq 1,3,7$. Also $a + d = Trace\, A = 0, \pm 2$. If a is odd then d is odd and b, c must be even in which case $A \equiv I \pmod 2$ contrary to assumption. Thus a and d are even and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2$. This shows that the elements x and y in the first part of the proof can be taken to be the usual basis elements and the last statement of the theorem follows.

Now we turn our attention to determine the possibility of existence of free actions of $G = S_n$ on spheres S^k or their manifold products $\underbrace{S^k \times S^k \times \ldots\ldots \times S^k}_{n\ copies} = \prod^n S^k$.

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