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FUZZY SETTING ON SUPRA σ- BAIRE SPACES

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ABSTRACT In this paper the concepts of fuzzy supra σ-Baire spaces are introduced and characterizations of fuzzy supra σ-Baire space are studied. Several examples are given to illustrate the concepts introduced in this paper.

KEYWORDS : Fuzzy supra dense set, fuzzy supra nowhere dense set, fuzzy supra F_σ-set, fuzzy supra G_δ-set, fuzzy supra σ-nowhere dense set, fuzzy supra σ-first category, fuzzy supra σ-second category, fuzzy supra σ-Baire space.

1.Introduction

The theory of fuzzy sets was initiated by L.A.Zadeh in his classical paper [8] in the year 1965 as an attempt to develop a mathematically precise framework in which to treat systems of phenomena which cannot themselves be characterized precisely. The potential of fuzzy notion was realized by the researchers and has successfully been applied for investigations in all the branches of Science and Technology. The paper of C.L.Chang [3] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological groups.

The concept of σ −Baire spaces in fuzzy setting was introduced and studied by the authors in [7]. M .E. Abd El- Monsef at al. [1] introduced Fuzzy supra topological spaces generalizing classical supra topological spaces introduced by A.S. Mashhour et al.[4]. In this paper, we introduce the concept of fuzzy supra σ −Baire space by using fuzzy supra σ −nowhere dense sets and investigate several characterizations and examples of fuzzy supra σ −Baire spaces.

2. Preliminaries

Definition 2.1 [5]: A fuzzy topology T on a set X is a family of fuzzy sets in X such that

(1) $0_X, 1_X \in T$;
(2) $A, B \in T \Rightarrow A \wedge B \in T$ and
(3) $A_i \in T \Rightarrow \vee A_i \in T$

The pair (X,T) is called a fuzzy topological space (FTS). The elements of T are called fuzzy open sets and the complement of fuzzy open set is called fuzzy closed set.

Definition 2.2 [2]: The closure and the interior of a fuzzy set A in FTS (X,T) are denoted and defined respectively as

$$Cl(A) = \wedge \{B: A \le B, B \text{ is a fuzzy closed set in } X\},$$

$$Int(A) = \vee \{B: B \le A, B \text{ is a fuzzy open set in } X\}.$$

Definition 2.3[1]: A collection T^* of fuzzy open sets in a set X is called a fuzzy supra topology on X if the following conditions are satisfied:

$$(1) 0_X, 1_X \in T \quad and \quad (2) A_i \in T \Rightarrow \vee A_i \in T$$

The pair (X,T^*) is called a fuzzy supra topological space (FSTS). The elements of T^* are called fuzzy supra open sets (FSOS) and the complement of a fuzzy supra open set is called fuzzy supra closed set (FSCS). The collection of all fuzzy supra open sets (resp.fuzzy supra closed sets) of the FSTS (X,T^*) is denoted by FSOS(X) (resp. FSCS(X)).

Remark 2.4 (1):

1. Every FTS is a FSTS.

2. If (X,T^*) is an associated FSTS with the FTS (X,T) (i.e. $T \subseteq T^*$), then every fuzzy open (closed) set in the FTS (X,T) is fuzzy supra open (closed) set in the FSTS (X,T^*).

Definition 2.5 [4]: Let (X,T^*) is a FSTS and A be fuzzy set in X, then the fuzzy supra closure and fuzzy supra interior are denoted and defined respectively as

$$Cl^*(A) = \wedge \{B : A \leq B, B \text{ is a fuzzy supra closed set in } X\},$$

$$Int^*(A) = \vee \{B : B \leq A, B \text{ is a fuzzy supra open set in } X\}.$$

Remark 2.6 [4]:

(1) The fuzzy supra closure of a fuzzy set A in a FSTS is the smallest fuzzy supra closed set containing A.
(2) The fuzzy supra interior of a fuzzy set A in a FSTS is the largest fuzzy supra open set containing in A.
(3) If (X,T^*) is an associated FSTS with the FTS (X,T) and A is any fuzzy set in X, then
$$Int(A) \leq int^*(A) \leq A \leq Cl^*(A) \leq Cl(A).$$

Properties of fuzzy supra closure and fuzzy supra interior which are needed in the sequel, are summarized in the following theorem.

Theorem 2.7 [4]: For any fuzzy sets A and B in a FSTS (X,T^*).

(1) $A \in FSCS(X) \Leftrightarrow Cl^*(A) = A,\ \ A \in FSOS(X) \Leftrightarrow Int^*(A) = A$;
(2) $A \leq B, \Rightarrow Cl^*(A) \leq Cl^*(B) and\ Int^*(A) \leq Int^*(B)$;
(3) $Cl^*(Cl^*(A)) = Cl^*(A)$ and $Int^*(Int^*(A)) = Int^*(A)$;
(4) $Cl^*(A \vee B) \geq Cl^*(A) \vee Cl^*(B)$;
(5) $Cl^*(A \wedge B) \leq Cl^*(A) \wedge Cl^*(B)$;
(6) $Int^*(A \vee B) \geq Int^*(A) \vee Int^*(B)$;
(7) $Int^*(A \wedge B) \leq Int^*(A) \wedge Int^*(B)$;
(8) $Cl^*(A^C) = (Int^*(A))^C, Int^*(A^C) = (Cl^*(A))^C$.

3. FUZZY SUPRA σ –NOWHERE DENSE SETS

Definition 3.1: A fuzzy set A in a FSTS (X,T^*) is called fuzzy supra dense if there exists no fuzzy supra closed set B in (X,T^*) such that $A < B < 1$. That is, $Cl^*(A) = 1$.

Definition 3.2: A fuzzy set A in a FSTS (X,T^*) is called fuzzy supra F_σ –set in (X,T^*) if $A = \vee_{i=1}^{\infty} A_i$, where $1 - A_i \in T^*$ for $i \in I$.

Definition 3.3: A fuzzy set A in a FSTS (X,T^*) is called a fuzzy supra G_δ –set in (X,T^*) if $A = \wedge_{i=1}^{\infty} A_i$, where $A_i \in T^*$ for $i \in I$.

Definition 3.4: A fuzzy set A in a FSTS (X,T^*) is called fuzzy supra nowhere dense if there exists no non-zero fuzzy open set B in (X,T^*) such that $B < cl^*(A)$. That is, $int^*cl^*(A) = 0$.

Definition 3.5: Let (X,T^*) be a fuzzy supra topological space. A fuzzy set A in (X,T^*) is called a fuzzy supra σ –nowhere dense set if A is a fuzzy supra F_σ –set in (X,T^*) such that $int^*(A) = 0$.

Example 3.6: Let $X = \{a,b\}$ be a set with a fuzzy supra topology (X,T^*). Then the fuzzy sets $A = \{a_{.6}, b_{.7}\}, B = \{a_{.7}, b_{.8}\}, c = \{a_{.6}, b_{.9}\}$. Then $T^* = \{0, A, B, C, A \vee B, B \vee C, B \wedge C, 1 - A, 1 - B, 1 - C, 1 - A \vee B, 1 - B \vee C, 1 - B \wedge C, 1\}$. Now, consider the fuzzy set $\alpha = (1 - A \vee B) \vee (1 - B \vee C) \vee (1 - B \wedge C)$ in (X,T^*). Then α is fuzzy supra F_σ –set in (X,T^*) and $int^*(A) = 0$ and hence α is a fuzzy supra σ – nowhere dense set in (X,T^*).

Proposition 3.7: In a fuzzy supra topological space (X,T^*) a fuzzy set A is fuzzy supra σ –nowhere dense set in (X,T^*) if and only if $1 - A$ is a fuzzy supra dense and fuzzy supra G_δ –set in (X,T^*).

Proof: Let A be a fuzzy supra σ –nowhere dense set in (X,T^*). Then $A = \vee_{i=1}^{\infty} A_i$, where $1 - A_i \in T^*$ for $i \in I$ and $int^*(A) = 0$. Then $1 - int^*(A) = 1 - 0 = 1$ implies that $cl^*(1 - A) = 1$. Also $(1 - A) = 1 - \vee_{i=1}^{\infty} (A_i) = \wedge_{i=1}^{\infty} (1 - A_i)$ where $(1 - A_i) \in T^*$, for $i \in I$. Hence we have $(1 - A)$ is a fuzzy supra dense and fuzzy supra G_δ –set in (X,T^*).

Conversely, let A be a fuzzy supra dense and fuzzy supra G_δ −set in (X, T^*). Then $A = \wedge_{i=1}^{\infty} (A_i)$ where $(A_i) \in T^*$, for $i \in I$. Now $(1 - A) = 1 - \wedge_{i=1}^{\infty} (A_i) = \vee_{i=1}^{\infty} (1 - A_i)$. Hence $1 - A$ is a fuzzy supra F_σ −set in (X, T^*) and $int^*(1 - A) = 1 - cl^*(A) = 1 - 1 = 0$.[Since A is a fuzzy supra dense]. Therefore $1 - A$ is a fuzzy supra σ −nowhere dense set in (X, T^*).

Proposition 3.8: If A is fuzzy supra dense set in (X, T^*) such that $B \leq (1 - A)$, where B is a fuzzy supra F_σ −set in (X, T^*), then B is a fuzzy supra σ −nowhere dense set in (X, T^*).

Proof: Let A be a fuzzy supra dense set in (X, T^*) such that $B \leq (1 - A)$. Now $B \leq (1 - A)$ implies that $int^*(B) \leq int^*(1 - A) = 1 - \quad cl^*(A) = 1 - 1 = 0$ and hence $int^*(B) = 0$. Therefore B is a fuzzy supra σ −nowhere dense set in (X, T^*).

Proposition 3.9: If A is a fuzzy supra F_σ −set and fuzzy supra nowhere dense set in (X, T^*), then A is a fuzzy supra σ −nowhere dense set in (X, T^*).

Proof: Now A≤ $cl^*(A)$ for any fuzzy set in (X, T^*). Then, $int^*(A) \leq int^*\, cl^*(A)$. Since A is a fuzzy supra nowhere dense set in $(X, T^*), int^*\, cl^*(A) = 0$ and hence $int^*(A) = 0$ and A is a fuzzy supra F_σ −set implies that A is a fuzzy supra σ −nowhere dense set in (X, T^*).

Definition 3.10: Let (X, T^*) be a fuzzy supra topological space. A fuzzy set A in (X, T^*) is called fuzzy supra σ −first category if $A = \vee_{i=1}^{\infty} (A_i)$ where (A_i)'s are fuzzy supra σ −nowhere dense sets in (X, T^*). Any other fuzzy set in (X, T^*) is said to be fuzzy supra σ −second category in (X, T^*).

Definition 3.11: Let A be a fuzzy supra σ −first category set in (X, T^*). Then $1 - A$ is called a fuzzy supra σ −residual set in (X, T^*).

Definition 3.12: A fuzzy supra topological space (X, T^*) is called fuzzy supra σ −first category if the fuzzy set 1_X is a fuzzy supra σ −first category space in (X, T^*). That is, $1_X = \vee_{i=1}^{\infty} (A_i)$ where (A_i)'s are fuzzy supra σ −nowhere dense sets in (X, T^*). Otherwise, (X, T^*) will be called a fuzzy supra σ −second category space.

4. FUZZY SUPRA σ −BAIRE SPACE

Definition 4.1: Let (X, T^*) be a fuzzy supra topological space. Then (X, T^*) is called a fuzzy supra σ −Baire Space if $int^*(\vee_{i=1}^{\infty} (A_i)) = 0$, where (A_i)'s are fuzzy supra σ −nowhere dense sets in (X, T^*).

Example 4.2: Let $X = \{a, b, c\}$ be a set with a fuzzy supra topology (X, T^*). Then the fuzzy sets $A = \{a_{.8}, b_{.6}, c_{.7}\}, B = \{a_{.6}, b_{.9}, c_8\}, c = \{a_{.7}, b_{.5}, c_{.9}\}$. Then $T^* = \{0, A, B, C, A \vee B, A \vee C, B \vee C, B \wedge C, 1 - A, 1 - B, 1 - C, 1 - A \vee B, 1 - B \vee C, 1 - B \wedge C, 1\}$. Now, consider the fuzzy set $\alpha = (1 - A \vee B) \vee (1 - B \vee C) \vee (1 - B \wedge C)$ in (X, T^*). Then α is fuzzy supra F_σ −set in (X, T^*) and $int^*(\alpha) = 0$ and hence α is a fuzzy supra $\sigma -$ nowhere dense set in (X, T^*).

$\beta = (1 - B) \vee (1 - A \vee C) \vee (1 - B \vee (A \wedge C))$ and $int^*(\beta) = 0$ and hence β is a fuzzy supra $\sigma -$ nowhere dense set in (X, T^*). Then α and β are fuzzy supra $\sigma -$ nowhere dense sets in (X, T^*) and also $int^*(\alpha \vee \beta) = 0$ and therefore (X, T^*) is a fuzzy supra σ −Baire Space.

Proposition 4.3: Let (X, T^*) be a fuzzy supra topological space. Then the following are equivalent:

1. (X, T^*) is a fuzzy supra σ −Baire Space.
2. $int^*(A) = 0$ for every fuzzy supra σ −first category set A in (X, T^*).
3. $cl^*(B) = 1$ for every fuzzy supra σ −residual set B in (X, T^*).

Proof: (1) ⇒ (2). Let A be a fuzzy supra σ −first category set in (X, T^*). Then $A = (\vee_{i=1}^{\infty} (A_i))$ where (A_i)'s are fuzzy supra σ −nowhere dense sets in (X, T^*). Then, we have $int^*(A) = int^*(\vee_{i=1}^{\infty} (A_i))$.

Since (X, T^*) is a fuzzy supra σ −Baire Space, $int^*(\vee_{i=1}^{\infty} (A_i)) = 0$. Hence $int^*(A) = 0$ for any fuzzy supra σ −first category set A in (X, T^*).

(2) ⇒ (3). Let B be a fuzzy supra σ −residual set in (X, T^*). Then $(1 - B)$ is a fuzzy supra σ −first category set A in (X, T^*). By hypothesis, $int^*(1 - B) = 0$. Then $1 - cl^*(B) = 0$. Hence $cl^*(B) = 1$ for any fuzzy supra σ −residual set B in (X, T^*).

(3) ⇒ (1). Let A be a fuzzy supra σ −first category set in (X, T^*). Then $A = (\vee_{i=1}^{\infty} (A_i))$ where (A_i)'s are fuzzy supra σ −nowhere dense sets in (X, T^*). Now A is a fuzzy supra σ −first category set in (X, T^*) implies that $(1 - A)$ is a fuzzy supra σ −residual set in (X, T^*). By hypothesis, we have $cl^*(1 - A) = 1$. Then $1 - int^*(A) = 1$. Hence $int^*(A) = 0$. That is, $int^*(\vee_{i=1}^{\infty}(A_i)) = 0$ where (A_i)'s are fuzzy supra σ −nowhere dense sets in (X,T^*). Hence (X,T^*) is a fuzzy supra σ −Baire Space.

Proposition 4.4: If the fuzzy supra topological space (X,T^*) is a fuzzy supra σ −Baire Space, then (X,T^*) is a fuzzy supra σ −second category space.

Proof: Let (X,T^*) is a fuzzy supra σ −Baire Space. Then $int^*(\vee_{i=1}^{\infty}(A_i)) = 0$ where (A_i)'s are fuzzy supra σ −nowhere dense set in (X,T^*). Then $\vee_{i=1}^{\infty}(A_i) \neq 1_X$ [otherwise, $\vee_{i=1}^{\infty}(A_i) = 1_X$ implies that $int^*(\vee_{i=1}^{\infty}(A_i)) = int^*1_X = 1_X$, which implies that $0 = 1$, a contradiction]. Hence (X,T^*) is a fuzzy supra σ −second category space.

Proposition 4.5: If the fuzzy supra topological space (X,T^*) . If $\wedge_{i=1}^{\infty}(A_i) \neq 0$, where (A_i)'s are fuzzy supra dense and fuzzy supra G_δ −set in (X,T^*), then (X,T^*) is a fuzzy supra σ −second category space.

Proof: Now $\wedge_{i=1}^{\infty}(A_i) \neq 0$ implies that $1 - (\wedge_{i=1}^{\infty}(A_i)) \neq 1 - 0 = 0$. Then we have $(\vee_{i=1}^{\infty}(1 - A_i)) \neq 1$. Since A_i is a fuzzy supra dense and fuzzy supra G_δ −set in (X,T^*), by proposition 3.1, $(1 - A_i)$ is a fuzzy supra σ −nowhere dense set in (X,T^*). Hence $(\vee_{i=1}^{\infty}(1 - A_i)) \neq 1$,where $(1 - A_i)$'s are fuzzy supra σ −nowhere dense set in (X,T^*). Hence (X,T^*) is not a fuzzy supra σ −first category space. Therefore (X,T^*) is a fuzzy supra σ −second category space.

Proposition 4.6: If the fuzzy supra topological space (X,T^*) is a fuzzy supra σ −first category space, then (X,T^*) is not a fuzzy supra σ −Baire Space.

Proof: Let the fuzzy supra topological space (X,T^*) is a fuzzy supra σ −first category space. Then $\vee_{i=1}^{\infty}(A_i)=1_X$, where (A_i)'s are fuzzy supra σ −nowhere dense sets in (X,T^*).Now $int^*(\vee_{i=1}^{\infty}(A_i))= int^*(1_X) = 1 \neq 0$. Hence by definition, (X,T^*) is not a fuzzy supra σ −Baire Space.

Proposition 4.7: If the fuzzy supra topological space (X,T^*) is a fuzzy supra σ −Baire Space and if $\vee_{i=1}^{\infty}(A_i)$=1, then there exists atleast one fuzzy supra F_σ −set A_i such that $int^*(A_i) \neq 0$.

Proof: Suppose that $int^*(A_i) = 0$, for $i = 1\ to\ \infty$, where $(A_i)'s$ are fuzzy supra σ −nowhere dense sets in (X,T^*). Then $\vee_{i=1}^{\infty}(A_i) = 1$. Implies that $int^*[\vee_{i=1}^{\infty}(A_i)] = int[1] = 1 \neq 0$, a contradiction to (X,T^*) being a fuzzy supra σ −Baire Space. Hence $int^*(A_i) \neq 0$, for atleast one fuzzy supra F_σ −set A_i in (X,T^*).

Proposition 4.8: If the fuzzy supra topological space (X,T^*) is a fuzzy supra σ −Baire Space, then no non-zero fuzzy supra open set is a fuzzy supra σ −first category set in (X,T^*).

Proof: Let A be a non-zero fuzzy supra open set in a fuzzy supra σ −Baire Space (X,T^*). Suppose that $A = \vee_{i=1}^{\infty}(A_i)$, where the fuzzy sets (A_i)'s are fuzzy supra σ −nowhere dense sets in (X,T^*). Then $int^*(A_i) = int^*(\vee_{i=1}^{\infty}(A_i))$. Since (X,T^*) is a fuzzy supra σ −Baire Space, $int^*(\vee_{i=1}^{\infty}(A_i)) = 0$. This implies that $int^*(A) = int^*(\vee_{i=1}^{\infty}(A_i))$. Since (X,T^*) is a fuzzy supra σ −Baire Space, $int^*(\vee_{i=1}^{\infty}(A_i)) = 0$. This implies that $int^*(A) = 0$. Then we will have $A = 0$, which is a contradiction, since $A \in T$ implies that $int^*(A) = A \neq 0$. Hence no non-zero fuzzy supra open set is a fuzzy supra σ −first category set in (X,T^*).

References

[1] Abd. El-Monsef M.F., and Ramadan A.E. On fuzzy supra topological spaces, Indian J.Pure. Appl.Math., 18(4) (1987) 322-329.
[2] K.K. Azad, On fuzzy semi continuity, Fuzzy almost continuity and fuzzy weskly continuity, J. Math. Anal.Appl., 82(1981), 14-32.
[3] Chang C.L., Fuzzy topological spaces, J. Math. Anal. And Appl. 24 (1968) 182-190.
[4] Mashhour A.S.Allan A.A., Mahmoud F.S and Khedr F.H., On supra topological space, Indian J.Pure Appl. Math. 14(4) (1983) 502-510.
[5] Palaniappan N., 2005, Fuzzy Topology, Narosa Publishing House.
[6] Srivastava M., Sinha P., Supra topological space and Hausdorffness, Int.J.Math.Archieve 6(8)(2015) 1-5.
[7] Thangaraj G., & Poongothai E., On fuzzy σ-Baire Spaces, Int.J.of Fuzzy Mathematics and systems, 3(4)(2013) 275-283.
[8] Zadeh L.A., Fuzzy Sets, Int. Contr. 8 (1965) 338-353.