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# A STUDY ON COMPARISON OF JACOBI AND GAUSS-SEIDEL ITERATIVE METHODS FOR THE SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

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**ABSTRACT** In this paper, two iterative methods of solving system of linear equations in eight variables has been compared and the methods considered are: Jacobi method and Gauss-Seidel method. The results show that Gauss-Seidel method is more efficient than Jacobi method by considering maximum number of iteration required to converge and accuracy.

**KEYWORDS** : Iterative methods; System of linear equations; Jacobi method; Gauss- Seidal method.

## INTRODUCTION

Numerical analysis is the area of mathematics and computer science that creates, analyses, and implements algorithms for solving numerically the problems of continuous mathematics[1].The development of numerical methods on a daily basis is to find the right solution techniques for solving problems in the field of applied science and pure science, such as weather forecasts, population, the spread of the disease, chemical reactions, physics, optics and others[2].

Collections of linear equations are called Systems of linear equations. They involve same set of variables. A linear equation in the variable $x_1, x_2, x_3, \ldots, x_n$ is any equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b_1 \qquad (1)$$

There are many approaches of solving system of linear equations i.e. direct methods and Indirect (iterative) methods.But there is no single method that is best for all situations. The direct methods give the exact solution in which there is no error except the round off error , whereas iterative methods give the approximate solutions in which there is some error[3]. In a direct method the amount of computation is fixed, while in an iterative method the amount of computation depends on the accuracy required. In general, one should prefer a direct method for the solution of linear system, but in the case of matrices with a large number of zero elements, it will be advantageous to use iterative methods .Iterative method, are very effective concerning computer storage and time requirements.

### Problem formulation

Here, we consider the system of 'n' linear equations in 'n' unknowns as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned} \qquad (2)$$

which may be represented as the matrix equation, AX=B **(3)**

Where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

In which the diagonal elements *aij* do not vanish. If this is not the case, then the equations should be rearranged, so that this condition is satisfied. Now we rewrite the system of equation (2) as

$$\begin{aligned} x_1 &= \frac{b_1-(a_{12}x_2+a_{13}x_3+\cdots+a_{1n}x_n)}{a_{11}} \\ x_2 &= \frac{b_2-(a_{21}x_1+a_{23}x_3+\cdots+a_{2n}x_n)}{a_{22}} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= \frac{b_n-(a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn-1}x_{n-1})}{a_{nn}} \end{aligned} \qquad (4)$$

Now, if an initial guess (zero iteration) $x_1^0, x_2^0, x_3^0, \ldots, x_n^0$ for all the unknowns is available, we can substitute these values into the right-hand side of the set of equations (4) and compute an updated guess (First iteration)for the unknowns, $x_1^1, x_2^1, x_3^1, \ldots, x_n^1$. Second iteration can be obtained by substituting first iteration in (4) and so on.

### Jacobi Method

The first iterative technique is called the Jacobi method, after Carl Gustav Jacob Jacobi (1804–1851), a German mathematician . This method is based on the assumption that the system has a unique solution and coefficient matrix A has no zero on its main diagonal.

In the Jacobi method, we starts with the initial guess $x_1^0, x_2^0, x_3^0, \ldots, x_n^0$ compute the next approximation of the solution as follows:

First iteration

$$\begin{aligned} x_1^1 &= \frac{b_1-(a_{12}x_2^0+a_{13}x_3^0+\cdots+a_{1n}x_n^0)}{a_{11}} \\ x_2^1 &= \frac{b_2-(a_{21}x_1^0+a_{23}x_3^0+\cdots+a_{2n}x_n^0)}{a_{22}} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n^1 &= \frac{b_n-(a_{n1}x_1^0+a_{n2}x_2^0+\cdots+a_{nn-1}x_{n-1}^0)}{a_{nn}} \end{aligned} \qquad (5)$$

After k iterations,

$$\begin{aligned} x_1^{k+1} &= \frac{b_1-(a_{12}x_2^k+a_{13}x_3^k+\cdots+a_{1n}x_n^k)}{a_{11}} \\ x_2^{k+1} &= \frac{b_2-(a_{21}x_1^k+a_{23}x_3^k+\cdots+a_{2n}x_n^k)}{a_{22}} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n^{k+1} &= \frac{b_n-(a_{n1}x_1^k+a_{n2}x_2^k+\cdots+a_{nn-1}x_{n-1}^k)}{a_{nn}} \end{aligned} \qquad (6)$$

### Gauss-Seidel Method

The next method is called Gauss-Seidel method, which is the modification of Jacobi method, named after Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). With the Jacobi method, the values of unknowns in the nth approximation remain unchanged until the entire nth approximation has been calculated. With the Gauss- Seidel method, on the other hand, we use the new values of each as soon as they are known.

Here, we starts with the initial guess $x_1^0, x_2^0, x_3^0, \ldots, x_n^0$ compute the next approximation of the solution as follows:

First iteration

$$\begin{aligned} x_1^1 &= \frac{b_1-(a_{12}x_2^0+a_{13}x_3^0+\cdots+a_{1n}x_n^0)}{a_{11}} \\ x_2^1 &= \frac{b_2-(a_{21}x_1^1+a_{23}x_3^0+\cdots+a_{2n}x_n^0)}{a_{22}} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n^1 &= \frac{b_n-(a_{n1}x_1^1+a_{n2}x_2^1+\cdots+a_{nn-1}x_{n-1}^1)}{a_{nn}} \end{aligned} \qquad (7)$$

After k iterations,

$$x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + a_{13}x_3^k + \cdots + a_{1n}x_n^k)}{a_{11}}$$

$$x_2^{k+1} = \frac{b_2 - (a_{21}x_1^{k+1} + a_{23}x_3^k + \cdots + a_{2n}x_n^k)}{a_{22}}$$

……………………………………………..

…………………………………………….. **(8)**

$$x_n^{k+1} = \frac{b_n - (a_{n1}x_1^{k+1} + a_{n2}x_2^{k+1} + \cdots + a_{nn-1}x_{n-1}^{k+1})}{a_{nn}}$$

### Convergence of Iterative Methods

The rate of convergence of iterative methods determine how fast the error $|x^k - x|$ goes to zero as k ,the number of iterations increases. Also,the iterative methods converge, for any choice of the first approximation $x_j^0$ (j=1,2,….) , if every equation of the system (2) satisfies the condition that the sum of the absolute values of the coefficients $a_{ij}/a_{ii}$ almost equal to, or in at least one equation less than unity.

(i.e) $|a_{ii}| \geq \Sigma|a_{ij}|$ (i=1,2,3…..n) ……… **(9)**

### Numerical Experiments

Here, we shall find out solution of same problem by Jacobi method & Gauss Seidal method . After that, we shall analyze their errors.

### Problem 01

Solve the equation using Jacobi's method.

$$5x_1 - x_2 - x_5 - x_8 = -2$$
$$-x_1 + 5x_2 - x_3 - x_7 - x_8 = -1$$
$$-x_2 + 5x_3 - x_4 - x_6 - x_7 = 4$$
$$x_1 - x_3 + 5x_4 - x_5 - x_8 = 13$$
$$-x_1 - x_2 + 5x_5 - x_6 + x_8 = 4$$
$$-x_3 - x_4 + 5x_6 + x_7 - x_8 = 2$$
$$-x_1 - x_5 + 5x_7 - x_8 = 9$$
$$-x_1 - x_3 - x_5 - x_7 + 5x_8 = 12$$

Taking the initial approximation $x_j^0 = 0 \; for \; j = 1,2,\dots,8$ Starting with these value and continuous to iterate we obtain the solution in the table below:

**Table 1. Iteration result for Jacobi method**

| Iteration | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 1 | -0.400000 | -0.200000 | 0.800000 | 2.600000 | 0.800000 | 0.400000 | 1.800000 | 2.400000 |
| 2 | 0.200000 | 0.720000 | 1.720000 | 3.480000 | 0.280000 | 1.200000 | 2.360000 | 3.000000 |
| 3 | 0.400000 | 1.256000 | 2.3520000 | 3.560000 | 0.624000 | 1.568000 | 2.496000 | 3.312000 |
| 4 | 0.638400 | 1.512000 | 2.5760000 | 3.7776000 | 0.782400 | 1.745600 | 2.667200 | 3.574400 |
| 5 | 0.773760 | 1.691200 | 2.740480 | 3.858880 | 0.864320 | 1.852160 | 2.799040 | 3.732800 |
| 6 | 0.857664 | 1.809216 | 2.840256 | 3.912768 | 0.916864 | 1.906624 | 2.874176 | 3.835520 |
| 7 | 0.912320 | 1.881523 | 2.900557 | 3.946995 | 0.947597 | 1.942874 | 2.922010 | 3.897792 |
| 8 | 0.945382 | 1.926536 | 2.938680 | 3.966725 | 0.967785 | 1.964667 | 2.951542 | 3.936496 |
| 9 | 0.966163 | 1.954420 | 2.961894 | 3.979516 | 0.980018 | 1.978072 | 2.969933 | 3.960678 |
| 10 | 0.979023 | 1.971734 | 2.976388 | 3.987285 | 0.987596 | 1.986431 | 2.981372 | 3.975602 |
| 11 | 0.986986 | 1.982477 | 2.985364 | 3.992112 | 0.992317 | 1.991581 | 2.988444 | 3.94876 |
| 12 | 0.991934 | 1.989134 | 2.990923 | 3.995114 | 0.995234 | 1.994782 | 2.992836 | 3.990622 |
| 13 | 0.994998 | 1.993263 | 2.994373 | 3.996969 | 0.997046 | 1.996765 | 2.995558 | 3.994185 |
| 14 | 0.996899 | 1.995823 | 2.996511 | 3.998121 | 0.998168 | 1.997994 | 2.997246 | 3.996395 |
| 15 | 0.998077 | 1.997410 | 2.997837 | 3.998835 | 0.998864 | 1.998756 | 2.998292 | 3.997765 |
| 16 | 0.998808 | 1.998394 | 2.998659 | 3.999278 | 0.999296 | 1.999229 | 2.998941 | 3.998614 |

Therefore, the solution obtained after sixteen successive iteration is $x_1$=0.998808 , $x_2$ =1.998394, $x_3$ = 2.998659, $x_4$ = 3.999278, $x_5$ = 0.999296, $x_6$ = 1.999229, $x_7$ = 2.998941, $x_8$ = 3.998614

### Problem 02

Solve the equation using Gauss-Seidel method.

$$5x_1 - x_2 - x_5 - x_8 = -2$$
$$-x_1 + 5x_2 - x_3 - x_7 - x_8 = -1$$
$$-x_2 + 5x_3 - x_4 - x_6 - x_7 = 4$$
$$x_1 - x_3 + 5x_4 - x_5 - x_8 = 13$$
$$-x_1 - x_2 + 5x_5 - x_6 + x_8 = 4$$
$$-x_3 - x_4 + 5x_6 + x_7 - x_8 = 2$$
$$-x_1 - x_5 + 5x_7 - x_8 = 9$$
$$-x_1 - x_3 - x_5 - x_7 + 5x_8 = 12$$

Taking the initial approximation $x_j^0 = 0 \; for \; j = 1,2,\dots,8$ Starting with these value and continuous to iterate we obtain the solution in the table below:

**Table 2. Iteration Result for Gauss-Seidel Method**

| Iteration | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 1 | -0.400000 | -0.280000 | 0.744000 | 2.828800 | 0.664000 | 1.114560 | 1.852800 | 2.972160 |
| 2 | 0.271232 | 0.968038 | 2.152840 | 3.703554 | 0.676334 | 1.795151 | 2.583945 | 3.536870 |
| 3 | 0.636249 | 1.581981 | 2.732926 | 3.861976 | 0.895302 | 1.909566 | 2.813684 | 3.815632 |
| 4 | 0.858583 | 1.844165 | 2.885878 | 3.947646 | 0.959336 | 1.967094 | 2.926710 | 3.926101 |
| 5 | 0.945921 | 1.936922 | 2.955675 | 3.979038 | 0.984767 | 1.986821 | 2971358 | 3.971544 |
| 6 | 0.978647 | 1.975445 | 2.982533 | 3.992039 | 0.993874 | 1.994952 | 2.988813 | 3.988773 |
| 7 | 0.991618 | 1.990347 | 2.993230 | 3.996852 | 0.997629 | 1.998008 | 2.995604 | 3.995616 |
| 8 | 0.996718 | 1.996234 | 2.997340 | 3.998773 | 0.999069 | 1.999225 | 2.998281 | 3.998281 |
| 9 | 0.998717 | 1.998524 | 2.998960 | 3.999519 | 0.999637 | 1.999696 | 2.999327 | 3.999328 |
| 10 | 0.999498 | 1.999423 | 2.999593 | 3.999812 | 0.999858 | 1.999881 | 2.999737 | 3.999737 |

Therefore, the solution oobtained after ten successive iteration is $x_1$ = 0.999498 , $x_2$ =1.999423, $x_3$ = 2.999593, $x_4$ = 3.998812, $x_5$ = 0.999858, $x_6$ = 1.999881, $x_7$ = 2.999737, $x_8$ = 3.999737

### Error analysis of Jacobi method

The true value of unknowns are are (1,2,3,4,1,2,3,4) while the computed value are (0.998808, 1.998394, 2.998659, 3.999278, 0.999296, 1.999229, 2.998941, 3.998614). Hence we determine the error as follows by using $X_8$

- **Absolute error =**|True value – approximate value|=|4-3.998614|=0.001386
- **Relative error:=** Absolute error/|True Value|=0.001386/4=0.0003465

- **Percentage relative error =** Relative error × 100% = 0.03465%

**Error analysis of Gauss-Seidel method**

The true value of unknowns are are (1,2,3,4,1,2,3,4) while the computed value are (0.999498, 1.999423,2.999593, 3.999812, 0.999858, 1.999881, 2.999737, 3.999737). Hence we determine the error as follows by using $x_8$ :

- **Absolute error =** |True value – approximate value|=|4-3.999737| =0.000263
- **Relative error: =** Absolute error/|True Value|=0.000263/ 4=0.00006575
- **Percentage relative error =** Relative error × 100% = 0.006575%

**Comparing percentage error of both methods**

| Methods | No of iteration | Error % |
|---|---|---|
| Jacobi method | 16 | 0.3465 |
| Gauss-Seidel method | 10 | 0.006575 |

## CONCLUSION

In this research paper, two iterative methods of solving system of linear equations have been presented . The numerical results and errors analysis of these two iterative methods for the system of linear equations showed that Gauss Seidal Method is more rapid in convergence than Jacobi method. Therefore, Gauss-Seidel method proved to be the best and effective in the sense that it converges very fast.

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