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ON THE CUBIC EQUATION WITH FOUR UNKNOWNS $x^3+y^3 = 57(h^2+3p^2)zw^2$

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ABSTRACT The homogeneous cubic equation with four unknowns represented by the Diophantine equation $x^3+y^3 = 57(h^2+3p^2)zw^2$ is analyzed for its patterns of non-zero integral solutions. A few interesting properties between the solutions and special numbers are presented.

KEYWORDS : Cubic equation with four unknowns, Integral solutions.

1. INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-15] for cubic equations with four unknowns. This communication concerns with yet another interesting equation $x^3+y^3 = {}^{57}(h^2+{}^3p^2)zw^2$ representing the homogeneous cubic equation with four unknowns for determining its infinitely many non-zero integral points. Also a few interesting properties are presented.

NOTATIONS:

1. $t_{m,n}$ - Polygonal number of rank n with size m
2. p_n^m - Pyramidal number of rank n with size m
3. gno_a - Gnomonic number of rank a.
4. so_n - Stella Octangular number of rank n
5. Pr_n - Pronic number of rank n.
6. $CP_{m,n}$ - Centered Pyramidal number of rank n with size m.

2. METHOD OF ANALYSIS:

The cubic Diophantine equation with four unknowns to be solved for getting non-zero integral solutions is

$$x^3 + y^3 = 57(h^2 + 3p^2)zw^2 \qquad \text{---(1)}$$

Introduction of the linear transformations

$$x = u + v, \; y = u - v \text{ and } z = 2u \qquad \text{---(2)}$$

in (1) leads to

$$u^2+3v^2=57(h^2+3p^2)zw^2 \qquad \text{---(3)}$$

Now, we solve (3) through different methods and thus obtain different patterns of solutions to (1).

Pattern 1:

Imagine that $w = w(a,b) = a^2+3b^2$ --- (4)

Where a and b are non-zero distinct integers Mark 57 as

$$57 = \frac{(3n+4ni\sqrt{3})(3n-4ni\sqrt{3})}{n^2} \qquad \text{--- (5)}$$

using (4) and (5) in (3) and applying the way of factorization, classify

$$u + i\sqrt{3}v = (3n + 4ni\sqrt{3})(h + i\sqrt{3}p)(a + i\sqrt{3}b)^2$$

Equating the real and imaginary parts, we have

$$u = h(3a^2 - 24ab - 9b^2) + p(-12a^2 - 18ab + 36b^2)$$

$$v = h(4a^2 + 6ab - 12b^2) + p(3a^2 - 24ab - 9b^2)$$

Hence in view of (2), the values of x, y, z are given by

$$x(h,p,a,b) = x = h(7a^2 - 18ab - 21b^2) + p(-9a^2 - 42ab + 21b^2)$$

$$y(h,p,a,b) = y = h(-a^2 - 30ab + 3b^2) + p(-15a^2 + 6ab + 45b^2)$$

$$z(h,p,a,b) = z = h(6a^2 - 48ab - 18b^2) + p(-24a^2 - 36ab + 72b^2) \qquad \text{(6)}$$

1) $7x(3,-1,a,2a^2-1) + y(3,-1,a,2a^2-1) + 5184\,Star_a + 158976FN_a{}^4 \equiv 0 \bmod (1104)$

Thus equation (6) represent the non-zero integral solutions to (1)

A few interesting properties observed are as follows:

1. $x(h,p,a,b) + y(h,p,a,b) - z(h,p,a,b) = 0$
2. $[x(h,p,a,b) + y(h,p,a,b)]^2 - z^2(h,p,a,b) = 0$
3. $x(1,1,a,1) + 7y(1,1,a,1) + 114\,Pro_a \equiv 0 \pmod{114}$
4. $z(h,p,a,1) - h\{6\,Pro_a - 27Gno_a + 9\} - p\{24\,Pro_a + 6Gno_a - 66\} = 0$

Pattern – II

Rewrite equation (3) as

$$u^2 + 3v^2 = 57(h^2 + 3p^2)zw^2 * 1 \qquad \text{--- (7)}$$

Mark 1 as

$$1 = \frac{(n+ni\sqrt{3})(n-ni\sqrt{3})}{(2n)^2} \qquad \text{--- (8)}$$

Using (4), (5) and (8) in (7) and applying the method of factorization. Classify,

$$u + i\sqrt{3}v = (n + ni\sqrt{3})(3n + 4ni\sqrt{3})(h + i\sqrt{3}P)(a + i\sqrt{3}b)^2 \qquad \text{--- (9)}$$

Equating the real and imaginary parts, we have

$$u = \tfrac{1}{2}\{h(-9^2 - 42ab + 27b^2) + p(-21a^2 + 54ab + 63b^2)\}$$

$$v = \tfrac{1}{2}\{h(7a^2 - 18ab - 21b^2) + p(-9a^2 - 42ab + 27b^2)\}$$

Hence in view of (2) the values of x, y, z are given by

$$x = \tfrac{1}{2}\{h(-2a^2 - 60ab + 6b^2) + p(-30a^2 + 12ab + 90b^2)\}$$

$$y = \tfrac{1}{2}\{h(-16a^2 - 24ab + 48b^2) + p(-12a^2 + 96ab + 36b^2)\}$$

$$z = h(-9a^2 - 42ab + 27b^2) + p(-21a^2 + 54ab + 63b^2)$$

$$w = a^2 + 3b^2$$

Our interest is to obtain the integer solution, so that the values of x and y are integers for suitable choices of the parameters a and b.

Put $a = 2A, b = 2B$

$$x(h,p,a,b) = x = h(-4A^2 - 120AB + 12B^2) + p(-60A^2 + 24AB + 180B^2)$$

$$y(h,p,a,b) = y = h(-32A^2 - 48AB + 96B^2) + p(-24A^2 + 192AB + 72B^2)$$

$$z(h,p,a,b) = z = h(-36A^2 - 168AB + 108B^2) + p(-84A^2 + 216AB + 252B^2)$$

$$w(h,p,a,b) = w = 4A^2 + 12B^2 \qquad \text{--- (10)}$$

Thus equation (10) represent the non-zero integral solutions to (1)

A few interesting properties observed are as follows:

1. $x(1,-1,2,b) + w(1,-1,2,b) + 156\,Ct_{2,a} \equiv 0 \pmod{132}$
2. $y(1,p,a,a+1) + 8w(1,p,a,a+1) - 48\,Pro_a(2p-1) + 24pT_{4,a} - 180Ct_{2,a} \equiv 0 \pmod{180}$
3. $z(2,1,a,a-1) + 624T_{4,a} + 20Star_a - 488\,CS_a = 20$
4. $x(1,1,a(a+1),a+2) + 14(Pro_a)^2 + 576\,TH_a - 18\,T_{4,a} \equiv 0 \pmod{36}$

Pattern – III

Rewrite equation (3) as

$$u^2 + 3v^2 = 19 * 3(h^2 + 3p^2)zw^2 \qquad \text{--- (11)}$$

Make 19 and 3 as

$$\left.\begin{array}{l} 19 = \frac{(4n+ni\sqrt{3})\,(4n-ni\sqrt{3})}{n^2} \\ 3 = \frac{(3n+ni\sqrt{3})\,(3n-ni\sqrt{3})}{(2n)^2} \end{array}\right\} \quad \text{--- (12)}$$

Using (4) and (12) in (11) and applying the method of factorization classify,

$$u + i\sqrt{3}v = (4n + ni\sqrt{3})\ (3n + ni\sqrt{3})\ \ (h + i\sqrt{3}P)\,(a + i\sqrt{3}b)^2 \quad \text{--- (13)}$$

Equating the real and imaginary parts, we have

$$u = \frac{1}{2}\{h(9a^2 - 42ab - 27b^2) + p\,(-21a^2 - 54ab + 62\,b^2)\}$$

$$v = \frac{1}{2}\{h\,(7a^2 + 18ab - 21b^2) + p\,(9a^2 - 42ab + 27b^2)\}$$

Hence in the view of (2), the values of x, y, z are given by

$$x = \frac{1}{2}\{h\,(16a^2 - 24ab - 48b^2) + p\,(-12a^2 - 96\,ab + 35b^2)\}$$

$$y = \frac{1}{2}\{h\,(2a^2 - 60ab - 6b^2) + p\,(-30a^2 - 12\,ab + 89b^2)\}$$

$$z = h\,(9a^2 - 42ab - 27b^2) + p\,(-21a^2 - 54ab + 62b^2)$$

$$w = a^2 + 3b^2$$

Our interest is to obtain the integer solution, so that the values of x and y are integers for suitable choices of the parameters a and b.

Put $a = 2A, b = 2B$

$$x\,(h,p,a,b) = x = h\,(32A^2 - 48AB - 96B^2) + p\,(-24A^2 - 192AB + 70B^2)$$
$$y\,(h,p,a,b) = y = h\,(4A^2 - 120AB - 12B^2) + p\,(-60A^2 - 24AB + 178B^2)$$
$$z\,(h,p,a,b) = z = h\,(36A^2 - 168AB - 108B^2) + p\,(-84A^2 - 216AB + 248B^2)$$
$$w\,(h,p,a,b) = w = 4A^2 + 12B^2 \quad \text{--- (14)}$$

Thus equal (14) represent non-zero integral solutions to (1)
A few interesting properties observed are as follows:

1) $x\,(3,2,a,2a^2+1) - 8y\,(3,2,a,2a^2+1) - 27760\,T_{4,a} - 82080H_a - 143424\,FN_a^4 = 2988$

2) $4y\,(5,-1,2a^2-1,a) - z\,(5,-1,2a^2-1,a) - 15552\,FN_a^4 + 900\,T_{4,a} = 324$

3) $z\,(1,-1,a(a+1),a+2) + 9w\,(1,-1,a\,(a+1),a+2) - 156\,(Pro_a)^2 - 288\,Pa^3 - 248T_{4,a} \equiv 0\ (mod\ 496)$

4) $w(na,na)$ is a square number, where $n = 1,2,\dots$

Pattern – IV

Rewrite (3) as

$$u^2 + 3v^2 = 19 * 3\,(h^2 + 3p^2)z\,w^2 * 1 \quad \text{--- (15)}$$

Using equation (4), (8) and (12) in (15) and applying the method of factorization, classify,

$$u + i\sqrt{3}v = (4n + ni\sqrt{3})(3n + ni\sqrt{3})(n + ni\sqrt{3})(h + i\sqrt{3}\,p)\,(a + i\sqrt{3}b)^2 \quad \text{--- (16)}$$

Equation the real and imaginary parts, we have

$$u = \frac{1}{4}\{h\,(-12a^2 - 36ab + 36b^2) + p\,(-48a^2 + 72\,ab + 144b^2)\}$$

$$v = \frac{1}{4}\{h\,(6a^2 - 24ab - 18b^2) + p\,(-12a^2 - 96\,ab + 36b^2)\}$$

Hence in the view of (2), the values of x, y, z are given by

$$x = \frac{1}{4}\{h\,(-6a^2 - 60ab + 18b^2) + p\,(-60a^2 - 24\,ab + 180b^2)\}$$

$$y = \frac{1}{4}\{h\,(-18a^2 - 12ab + 54b^2) + p\,(-36a^2 + 168\,ab + 108b^2)\}$$

$$z = \frac{1}{4}\{h\,(-24a^2 - 72ab + 72b^2) + p\,(-96a^2 + 144\,ab + 288b^2)\}$$

Our interest is to obtain the integer solution, so that the values of x and y are integers for suitable choices of the parameters a and b.

Put $a = 4A, b = 4B$

$$x\,(h,p,a,b) = x = h\,(-24A^2 - 240AB + 72B^2) + p\,(-240A^2 - 96\,AB + 720B^2)$$
$$y\,(h,p,a,b) = y = h\,(-72A^2 - 48AB + 216B^2) + p\,(-144A^2 + 672\,AB + 432B^2)$$
$$z\,(h,p,a,b) = z = h\,(-96A^2 - 288AB + 288B^2) + p\,(-384A^2 + 576\,AB + 1152B^2)$$
$$w\,(h,p,a,b) = w = 16A^2 + 48B^2 \quad \text{--- (17)}$$

Thus equal (17) represents the non-zero integral solutions to (1)

A few interesting properties observed are as follows:

2) $z\,(1,-1,3,2) + 6\,w\,(1,-1,3,2)$ is a Nasty Number

3) $w(na,na)$ is a square number, where $n = 1,2,\dots$

4) $x\,(1,2,\ 2a^2,a) + 252\,(Ct_{2,a} * T_{18,a}) + 840\,P_a^5 \equiv -300\ (mod\ 1464)$

3. Conclusion:

To conclude, one may search for other patterns of solutions and their corresponding properties.

4. REFERENCE:

1) L.E. Dicson, History of Theory of numbers, Vol.2, Diophantine analysis, New Yark, Dover, 2005.
2) L.J. Mordell, Diophantine Equations, Academic Press, New York, 1969.
3) Carmichael. R.D, The Theory of numbers and Diophantine Analysis, New York, Dover, 1959.
4) M.A. Gopalan and S. Premalatha, Integral solutions of $(x+y)(xy+w^2)=2\ (k^2+1)\ z^3$, Bulletin of pure and applied sciences, Vol. 29E(No.2), p_p 197-202, 2009.
5) M.A. Gopalan and V. PandiChelvi, Remarkable solutions on the cubic equation with four unknows $x^3+y^3+z^3=28(x+y+z)w^2$, Antarctica J. of maths, Vol. 4, No.4, p_p 393-401, 2010.
6) M.A. Gopalan and B. Sivagami, Integral solutions of homogeneous cubic equation with four unknows $x^3+y^3+z^3=3xyz+2(x+y)w^3$, Impact. J. Sci.Tech, Vol.4, No. p_p 53-60, 2010.
7) M.A. Gopalan and S. Pramelatha, on the cubic Diophantine equation with four unknows $(x-y)(xy-w^2)=2(n2+2n)z^3$, International Journal of mathematical sciences, Vol.9, No.1-2, Jan-June, p_p 171-175, 2010.
8) M.A. Gopalan and J. Kaligarani, Integral solutions of $x^3+y^3+(x+y)xy=z^3+w^3+(z+w)zw$, Bulletin of pure and applied sciences, Vol.29E(No.1), p_p 169-173, 2010.
9) M.A. Gopalan and S. Pramelatha, Integral solutions of $(x+y)(xy+w^2)=2(k+1)z^3$, The Global Journal of applied mathematics and Mathematical sciences, Vol.3, No.1-2, p_p 51-55, 2010.
10) M.A. Gopalan and S. Vidhyalakshmi and S. Mallika, Obsertvation on the cubic equation with four unknows $^2(x^3+y^3)=z^3+w2\,(x+y)$, IJAMP Vol.4, No.2, p_p 103-107, Jul-Dec 2012.
11) Anbuselvi R, Kannaki K, On ternary Quadratic Equation $11x^2+3y^2=14z^2$ Volume 5, Issue 2, Feb 2016, Pg No. 65-68.
12) Anbuselvi R, Kannaki K, On ternary Quadratic Equation $x^2+xy+y^2=12z^2$ IJAR 2016: 2 (3); 533-535.
13) Anbuselvi R, Kannaki K, On ternary Quadratic Equation $3(x^2+y^2)-5xy+x+y+1=15z^3$ IJSR Sep 2016: 5(9); 42-48.
14) Anbuselvi R, Kannaki K, On ternary Quadratic Diophantine Equation $7(x^2+y^2)-13xy+x+y+1=31z^2$ IERJ Feb 2017: 3(2); 52-57.
15) Anbuselvi R, Kannaki K, On the Homogeneous Biquadratic Equation with Four Unknowns $x^4+y^4+z^4=98w^4$ GJRA Oct 2017 6(10); 92-93.