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Original Research Paper

Mathematics

ON ASYMPTOTIC AND OSCILLATORY BEHAVIOR OF THIRD ORDER FUNCTIONAL NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we study the third-order functional nonlinear di€erential equation of the form as $\{\alpha_2(t)\,ø\alpha_2\,[(\alpha_1(t)\,ø\,\alpha_1\,(y'(t)))']\} + q(t)ø\alpha\,(y(g(t))) = 0;$ where $ø_\gamma(u) := |u|^{\gamma-1}\,u$. Results are obtained for the asymptotic and oscillatory behavior of the solutions. This work extends and improves some known results in the literature on third order nonlinear differential equations.

KEYWORDS :

1 Introduction

We are concerned with the asymptotic and oscillatory behavior of the third order nonlinear functional differential equation

$$\left\{a_2(t)\phi_{\alpha_2}\left[\left(a_1(t)\phi_{\alpha_1}\left(y'(t)\right)\right)'\right]\right\}' + q(t)\phi_\alpha\left(y(g(t))\right) = 0, \quad (1.1)$$

where $\phi_\alpha(u) := |u|^{\alpha-1}u$. Throughout this paper we make the following assumptions:

(A1) $\alpha_i, \alpha > 0, i = 1,2$, are constants, and $a_1(t)$ and $a_2(t)$ are positive continuous functions on $\mathbb{T}$;

(A2) $q(t)$ are nonnegative continuous functions on $\mathbb{R}$ with $q(t) \neq 0$;

(A3) $g(t): [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions such that $\lim_{t\to\infty} g(t) = \infty$.

By a solution of Eq. (1.1) we mean a nontrivial real-valued function $y(t) \in C^1([t_x,\infty), \mathbb{R})$ for some $t_x \geq 0$ which has the property that and satisfies Eq. (1.1) on $[t_x, \infty)$, such that $a_1(t)\phi_{\alpha_1}\left(y'(t)\right) \in C^1[t_x,\infty)$, $a_2(t)\phi_{\alpha_2}\left[\left(a_1(t)\phi_{\alpha_1}\left(y'(t)\right)\right)'\right] \in C^1[t_x,\infty)$ and $y(t)$ satisfies Eq. (1.1) on $[t_x,\infty)$ and satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for any $T \geq t_x$. A solution of equation (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. In the sequel it will be always assumed that equation (1.1) has nontrivial solutions which exist for all $t_0 \geq 0$. Equation (1.1) is called oscillatory if all solutions are oscillatory. In the last few years, the oscillation theory and asymptotic behavior of differential equations and their applications have received more and more attentions, the reader is referred to the papers [1]-[18] and the references cited there in. Our aim is to investigate the oscillatory criteria for all solutions of equation (1.1) with the cases, for $k = 1,2$

$$\int_{t_0}^{\infty} a_k^{-\frac{1}{\alpha_k}}(t)dt = \infty,$$

and

$$\int_{t_0}^{\infty} a_k^{-\frac{1}{\alpha_k}}(t)dt < \infty.$$

Our work is motivated by the papers [7, 9], [12]. In fact Grace *et al* [12] establish oscillation criteria for the third order nonlinear differential equation of the form

$$\left(a(t)\left(x''(t)\right)^{\alpha}\right)' + q(t)f(x(g(t))) = 0. \quad (1.2)$$

Recently, Baculikova and Džurina [7] provide general classification of oscillatory and asymptotic behaviors of equation (1.2). We introduce the following notations which will be needed throughout

the study: $\alpha := \alpha_1\alpha_2$

$$L_i(y(t)) := a_i(t)\phi_{\alpha_i}\left[\left(L_{i-1}(y(t))\right)'\right], i = 1,2 \text{ with } L_0(y(t)) := y(t),$$

$$A_1(t) := \int t\, a_2^{-\frac{1}{\alpha_2}}(s)ds;$$

$$A_2(t) := \int t\, a_1^{-\frac{1}{\alpha_1}}(s)A_1^{\frac{1}{\alpha_1}}(s)ds;$$

$$Q_1(t) := q(t)\left[\int_{g(t)}^{\infty} a_1^{-\frac{1}{\alpha_1}}(s)ds\right]^{\alpha}$$

$$Q_2(t) := q(t)\left(\int g(t)\, a_1^{-\frac{1}{\alpha_1}}(s)ds\right)^{\alpha}\left(\int_{g(t)}^{\infty} a_2^{-\frac{1}{\alpha_2}}(s)ds\right)^{\frac{\alpha}{\alpha_1}}$$

and

$$\tau^{-1}(t) := \min\{s \in \mathbb{T} : \tau(s) \geq t+1\} \text{ with } \tau(t) := \min\{t, g(t)\}, \quad (1.3)$$

2 Main Results

We start with the following lemmas which will play an important role in the proofs of our results. The proof of first two lemmas is straightforward; thus, we omit the details.

Lemma 2.1 *If $y(t)$ is an eventually positive solution of the Eq. (1.1), then $\left[L_1(y(t))\right]'$ and $y'(t)$ are eventually of one sign.*

Lemma 2.2 *If $y(t)$ is an eventually positive solution of the Eq. (1.1) and corresponding $y(t)$ satisfies*

$$y'(t) < 0 \quad \text{and} \quad [L_1(y(t))]' > 0$$

eventually, then $y(t)$ tends to nonnegative finite limit eventually.

Lemma 2.3 *If $y(t)$ is an eventually positive solution of the Eq. (1.1) and corresponding $y(t)$ satisfies*

$$y'(t) > 0 \quad \text{and} \quad [L_1(y(t))]' > 0$$

eventually, then

$$y(t) > A_2(t)\phi_\alpha^{-1}[L_2(y(t))]; \quad (2.1)$$

where $\alpha := \alpha_1\alpha_2$ and for $t \in (t_0,\infty)$,

$$\left[\frac{y(t)}{A_2(t)}\right]' < 0. \quad (2.2)$$

Proof. Since $y(t)$ is eventually positive solution of Eq. (1.1), without loss of generality, assume $y(t) > 0$ and $y(g(t)) > 0$ on $[t_0,\infty)$. Then, from $L_2(y(t))$ is strictly decreasing on $[t_0,\infty)$,

$$L_1(y(t)) > L_1(y(t)) - L_1(y(t_0)) = \int_{t_0}^{t} a_2^{-\frac{1}{\alpha_2}}(s)\phi_{\alpha_2}^{-1}[L_2(y(s))]\,ds$$

$$\geq \phi_{\alpha_2}^{-1}[L_2(y(t))]\int_{t_0}^{t} a_2^{-\frac{1}{\alpha_2}}(s)ds = \phi_{\alpha_2}^{-1}[L_2(y(t))]A_1(t), \quad (2.3)$$

which implies that

$$y'(t) > a_1^{-\frac{1}{\alpha_1}}(t)A_1^{\frac{1}{\alpha_1}}(t)\phi_\alpha^{-1}[L_2(y(t))],$$

where $\alpha = \alpha_1\alpha_2$. In the same way, we have

$$y(t) > \phi_\alpha^{-1}[L_2(y(t))]\int_{t_0}^{t} a_1^{-\frac{1}{\alpha_1}}(s)A_1^{\frac{1}{\alpha_1}}(s)ds = A_2(t)\phi_\alpha^{-1}[L_2(y(t))].$$

We note that

$$\left(\frac{L_1(y(t))}{A_1(t)}\right)' = \frac{1}{A_1^2(t)}\left(\left(L_2(y(t))\right)' A_1(t) - L_1(y(t))A_1'(t)\right)$$

$$= \frac{A_1'(t)}{A_1^2(t)}\left(\phi_{\alpha_2}^{-1}[L_2(y(t))]\,A_1(t) - L_1(y(t))\right),$$

so by (2.3), we get

$$\left(\frac{L_1(y(t))}{A_1(t)}\right)' < 0 \quad \text{for}\, t \in (t_0, \infty).$$

Then

$$y(t) > y(t) - y(t_0) = \int_{t_0}^{t} \phi_{\alpha_1}^{-1}[L_1(y(s))]a_1^{-\frac{1}{\alpha_1}}(s)ds$$

$$= \int_{t_0}^{t} \phi_{\alpha_1}^{-1}\left[\frac{L_1(y(s))}{A_1(s)}\right] a_1^{-\frac{1}{\alpha_1}}(s)A_1^{\frac{1}{\alpha_1}}(s)ds$$

$$\geq \phi_{\alpha_1}^{-1}\left[\frac{L_1(y(t))}{A_1(t)}\right]\int_{t_0}^{t} a_1^{-\frac{1}{\alpha_1}}(s)A_1^{\frac{1}{\alpha_1}}(s)ds$$

$$= A_2(t)\phi_{\alpha_1}^{-1}\left[\frac{L_1(y(t))}{A_1(t)}\right]. \tag{2.4}$$

Also

$$\left[\frac{y(t)}{A_2(t)}\right]' = \frac{1}{A_2^2(t)}\left(y'(t)A_2(t) - y(t)A_2'(t)\right)$$

$$= \frac{A_2'(t)}{A_2^2(t)}\left(\phi_{\alpha_1}^{-1}\left[\frac{L_1(y(t))}{A_1(t)}\right]A_2(t) - y(t)\right),$$

so by (2.4), we see that

$$\left[\frac{y(t)}{A_2(t)}\right]' < 0 \quad \text{for}\, t \in (t_0, \infty).$$

This completes the proof.

Lemma 2.4 *Suppose that*

(B1)

$$\int_{t_0}^{\infty} a_1^{-\frac{1}{\alpha_1}}(s)ds = \infty, \tag{2.5}$$

or

$$\int_{t_0}^{\infty} a_1^{-\frac{1}{\alpha_1}}(s)ds < \infty, \tag{2.6}$$

and

$$\int_{t_0}^{\infty} a_1^{-\frac{1}{\alpha_1}}(s)\{\int_{t_0}^{s} a_2^{-\frac{1}{\alpha_2}}(v)\left[\int_{t_0}^{v} Q_1(w)\, dw\right]^{\frac{1}{\alpha_2}} dv\}^{\frac{1}{\alpha_1}} ds = \infty. \tag{2.7}$$

(B2)

$$\int_{t_0}^{\infty} a_2^{-\frac{1}{\alpha_2}}(s)ds = \infty, \tag{2.8}$$

or

$$\int_{t_0}^{\infty} a_2^{-\frac{1}{\alpha_2}}(s)ds < \infty, \tag{2.9}$$

and

$$\int_{t_0}^{\infty} a_2^{-\frac{1}{\alpha_2}}(v)\left[\int_{t_0}^{v} Q_2(w)dw\right]^{\frac{1}{\alpha_2}} dv = \infty. \tag{2.10}$$

If $y(t)$ is an eventually positive solution of the Eq. (1.1), then $[L_1(y(t))]'$ is eventually positive.

Proof. As in the proof of Lemma 2.1, we have $L_2(y(t))$ is nonincreasing on $[t_0, \infty)$, and $[L_1(y(t))]'$ and $y'(t)$ are eventually of one sign. We show that $[L_1(y(t))]'$ is eventually positive. Otherwise, it is eventually negative. We consider the following two cases:

(a) $y'(t) < 0$ and $[L_1(y(t))]' < 0$ eventually. In this case, there exists $t_1 \geq t$ such that

$$y'(t) < 0 \text{and} [L_1(y(t))]' < 0 \quad \text{for } t \geq t_1.$$

Let (2.5) holds. Since $L_1(y(t))$ is strictly decreasing on $[t_1, \infty)$, we obtain

$$y(t) = y(t_1) + \int_{t_1}^{t} \phi_{\alpha_1}^{-1}[L_1(y(s))]a_1^{-\frac{1}{\alpha_1}}(s)ds$$

$$< y(t_1) + \phi_{\alpha_1}^{-1}[L_1(y(t_1))]\int_{t_1}^{t} a_1^{-\frac{1}{\alpha_1}}(s)ds.$$

By Eq. (2.5), we have $\lim_{t\to\infty} y(t) = -\infty$, which contradicts that $y(t) > 0$ is a positive solution of Eq. (1.1).

Let (2.6) and (2.7) hold. Let $t_2 \in t_1, \infty)$ such that $g(t) \geq t_1$ for $t \geq t_2$. Then for $t \geq t_2$,

$$y(g(t)) > -\int_{g(t)}^{\infty} \phi_{\alpha_1}^{-1}[L_1(y(t_1))]a_1^{-\frac{1}{\alpha_1}}(s)ds$$

$$> -\phi_{\alpha_1}^{-1}\left[L_1\left(y(g(t))\right)\right]\int_{g(t)}^{\infty} a_1^{-\frac{1}{\alpha_1}}(s)ds$$

$$> -\phi_{\alpha_1}^{-1}[L_1(y(t_1))]\int_{g(t)}^{\infty} a_1^{-\frac{1}{\alpha_1}}(s)ds.$$

Therefore,

$$y^\alpha(g(t)) > L\left(\int_{g(t)}^{\infty} a_1^{-\frac{1}{\alpha_1}}(s)ds\right)^{\alpha} \quad \text{for} t \geq t_2, \tag{2.11}$$

where $L := \{-\phi_{\alpha_1}^{-1}[L_1(y(t_1))]\}^{\alpha} > 0$. From Eq. (1.1) and (2.11), we have $\{L_2(y(t))\}' < -LQ_1(t)$. Integrating from t_2 to t, we see

$$L_2(y(t)) < L_2(y(t)) - L_2(y(t_2)) < -L\int_{t_2}^{t} Q_1(w)dw,$$

which implies that

$$[L_1(y(t))]' < -L^{\frac{1}{\alpha_2}}a_2^{-\frac{1}{\alpha_2}}(t)\left(\int_{t_2}^{t} Q_1(w)dw\right)^{\frac{1}{\alpha_2}}.$$

Again, integrating the above inequality from t_2 to t, we get

$$L_1(y(t)) < L_1(y(t)) - L_1(y(t_2)) < -L^{\frac{1}{\alpha_2}}\int_{t_2}^{t} a_2^{-\frac{1}{\alpha_2}}(v)\left(\int_{t_2}^{v} Q_1(w)dw\right)^{\frac{1}{\alpha_2}} dv,$$

which yields

$$y(t)-y(t_2)<-L^{\frac{1}{\alpha}}\int_{t_2}^{t}a_1^{-\frac{1}{\alpha_1}}(s)\{\int_{t_2}^{s}a_2^{-\frac{1}{\alpha_2}}(v)\left[\int_{t_2}^{v}\int_{a}^{b}Q_1(w)dw\right]^{\frac{1}{\alpha_2}}dv\}^{\frac{1}{\alpha_1}}ds.$$

From Eq. (2.7), we have $\lim_{t\to\infty}y(t)=-\infty$, which contradicts the fact that $y(t)$ is a positive solution of Eq. (1.1).

(b) $y'(t)>0$ and $[L_1(y(t))]'<0$ eventually. In this case, there exists $t_1\geq t$ such that

$$y'(t)>0 \text{ and } [L_1(y(t))]'<0 \quad \text{for } t\geq t_1.$$

Assume (2.8) holds. Since $L_2(y(t))$ is nonincreasing on $[t_1,\infty)$, we get

$$L_1(y(t))-L_1(y(t_1))=\int_{t_1}^{t}\phi_{\alpha_2}^{-1}[L_2(y(s))]a_2^{-\frac{1}{\alpha_2}}(s)ds$$

$$<\phi_{\alpha_2}^{-1}[L_2(y(t_1))]\int_{t_1}^{t}a_2^{-\frac{1}{\alpha_2}}(s)ds.$$

By (2.8), we have $\lim_{t\to\infty}L_1(y(t))=-\infty$, which contradicts that $y'(t)>0$ for $t\geq t_1$.

Assume (2.9) and (2.10) hold. Again, let $t_2\in t_1,\infty)$ such that $g(t)\geq t_1$ for $t\geq t_2$. Then for $t\geq t_2$,

$$y(g(t))>y(g(t))-y(t_1)=\int_{t_1}^{g(t)}\phi_{\alpha_1}^{-1}[L_1(y(s))]a_1^{-\frac{1}{\alpha_1}}(s)ds$$

$$>\phi_{\alpha_1}^{-1}\left[L_1\left(y(g(t))\right)\right]\int_{t_1}^{g(t)}a_1^{-\frac{1}{\alpha_1}}(s)ds, \tag{2.12}$$

and

$$L_1\left(y(g(t))\right)>-\int_{g(t)}^{\infty}\phi_{\alpha_2}^{-1}[L_2(y(s))]a_2^{-\frac{1}{\alpha_2}}(s)ds$$

$$>-\phi_{\alpha_2}^{-1}\left[L_2\left(y(g(t))\right)\right]\int_{g(t)}^{\infty}a_2^{-\frac{1}{\alpha_2}}(s)ds$$

$$>-\phi_{\alpha_2}^{-1}[L_2(y(t_1))]\int_{g(t)}^{\infty}a_2^{-\frac{1}{\alpha_2}}(s)ds. \tag{2.13}$$

Substituting (2.13) into (2.12), we get that for $t\geq t_2$,

$$y(g(t))>-\phi_{\alpha}^{-1}[L_2(y(t_1))]\int_{t_1}^{g(t)}a_1^{-\frac{1}{\alpha_1}}(s)ds\left(\int_{g(t)}^{\infty}a_2^{-\frac{1}{\alpha_2}}(s)ds\right)^{\frac{1}{\alpha_1}},$$

and hence

$$y^{\alpha}(g(t))>L\left(\int_{t_1}^{g(t)}a_1^{-\frac{1}{\alpha_1}}(s)ds\right)^{\alpha}\left(\int_{g(t)}^{\infty}a_2^{-\frac{1}{\alpha_2}}(s)ds\right)^{\frac{\alpha}{\alpha_1}}, \tag{2.14}$$

where $L:=\left(-\phi_{\alpha}^{-1}[L_2(y(t_1))]\right)^{\alpha}>0$. By Eq. (1.1) and (2.14), we get $[L_2(y(t))]'<-LA_2(t)$ .Integrating both sides from t_2 to t, we have

$$L_2(y(t))<L_2(y(t))-L_2(y(t_2))<-L\int_{t_2}^{t}Q_2(w)dw,$$

which implies that

$$[L_2(y(t))]'<-L^{\frac{1}{\alpha_2}}r_2^{-\frac{1}{\alpha_2}}(t)\left[\int_{t_2}^{t}Q_2(w)dw\right]^{\frac{1}{\alpha_2}}.$$

Again, integrating both sides from t_2 to t, we get

$$-L_1(y(t_2))<L_1(y(t))-L_1(y(t_2))$$

$$<-L^{\frac{1}{\alpha_2}}\int_{t_2}^{t}r_2^{-\frac{1}{\alpha_2}}(v)\left[\int_{t_2}^{v}Q_2(w)dw\right]^{\frac{1}{\alpha_2}}dv$$

$$<-L^{\frac{1}{\alpha_2}}\int_{t_2}^{t}r_2^{-\frac{1}{\alpha_2}}(v)\left[\int_{t_2}^{v}Q_2(w)dw\right]^{\frac{1}{\alpha_2}}dv,$$

which contradicts (2.10). This completes the proof.

Lemma 2.5 *Suppose that*

(B3) either

$$\int_{t_0}^{\infty}q(s)ds=\infty; \tag{2.15}$$

$$\int_{t_0}^{\infty}a_2^{-\frac{1}{\alpha_2}}(s)\left(\int_{s}^{\infty}q(v)dv\right)^{\frac{1}{\alpha_2}}ds=\infty; \tag{2.16}$$

or

$$\int_{t_0}^{\infty}a_1^{-\frac{1}{\alpha_1}}(s)\left[\int_{s}^{\infty}a_2^{-\frac{1}{\alpha_2}}(v)\left(\int_{v}^{\infty}q(w)dw\right)^{\frac{1}{\alpha_2}}dv\right]^{\frac{1}{\alpha_1}}ds=\infty. \tag{2.17}$$

If $y(t)$ is an eventually positive solution of the Eq. (1.1) and corresponding $y(t)$ satisfies

$$y'(t)<0 \quad \text{and} \quad [L_1(y(t))]'>0$$

eventually, then $y(t)$ tends to zero eventually.

Proof. Since $y(t)$ is eventually positive solution of Eq. (1.1), without loss of generality, assume $y(t)>0$ and $y(g(t))>0$ on $[t_0,\infty)$. We show that if $y'(t)$ is eventually negative, then $y(t)$ tends to zero eventually. In this case, $y'(t)<0$ eventually. Hence

$$\lim_{t\to\infty}y(t)=k_1\geq 0 \text{ and } \lim_{t\to\infty}L_1(y(t))=k_2\leq 0.$$

Assume $k_1>0$. Hence

$$y^{\alpha}(g(t))>k_1^{\alpha}=:k>0 \quad \text{for } t\in[t_1,\infty).$$

Integrating Eq. (1.1) from t to $\eta\in t,\infty)$ and noting that $L_2(y(t))>0$ eventually, we obtain

$$L_2(y(t))>-L_2(y(\eta))+L_2(y(t))>\int_{t}^{\eta}q(w)y^{\alpha}(g(w))dw>k\int_{t}^{\eta}q(w)dw.$$

Hence by taking limits as $\eta\to\infty$ we have

$$L_2(y(t))>k\int_{t}^{\infty}q(w)dw.$$

If (2.15) holds, we have reached a contradiction. Otherwise,

$$[L_1(y(t))]'>k^{\frac{1}{\alpha_2}}a_2^{-\frac{1}{\alpha_2}}(t)\left(\int_{t}^{\infty}q(w)dw\right)^{\frac{1}{\alpha_2}}.$$

Again, integrating this inequality from t to ∞ and noting that $L_1(y(t))\leq 0$ eventually, we get

$$-L_1(y(t))>k^{\frac{1}{\alpha_2}}\int_{t}^{\infty}a_2^{-\frac{1}{\alpha_2}}(v)\left(\int_{t}^{\infty}q(w)dw\right)^{\frac{1}{\alpha_2}}dv.$$

If (2.16) holds, we have reached a contradiction. Otherwise,

$$-y'(t)>k^{\frac{1}{\alpha}}a_1^{-\frac{1}{\alpha_1}}(t)\left[\int_{t}^{\infty}a_2^{-\frac{1}{\alpha_2}}(v)\left(\int_{t}^{\infty}q(w)dw\right)^{\frac{1}{\alpha_2}}dv\right]^{\frac{1}{\alpha_1}},$$

where $\alpha=\alpha_1\alpha_2$. Finally, integrating the last inequality from t_1 to t, we get

$$-y(t)+y(t_1)>k^{\frac{1}{\alpha}}\int_{t_1}^{t}a_1^{-\frac{1}{\alpha_1}}(s)\left[\int_{t}^{\infty}a_2^{-\frac{1}{\alpha_2}}(v)\left(\int_{t}^{\infty}q(w)dw\right)^{\frac{1}{\alpha_2}}dv\right]^{\frac{1}{\alpha_1}}ds.$$

Hence by (2.17), we have $\lim_{t\to\infty}y(t)=-\infty$, which contradicts the fact that $y(t)$ is a positive solution of Eq. (1.1). This shows that $\lim_{t\to\infty}y(t)=0$ and hence completes the proof.

Next, we establish some oscillation criteria of solutions to Eq. (1.1).

Theorem 2.1 *Let* (B_1) *and* (B_2) *hold. Assume for sufficiently large* $T\in t_0,\infty)$,

$$\int_{T}^{\infty}q(s)ds=\infty. \tag{2.18}$$

Then every solution of Eq. (1.1) is either oscillatory or tends to zero eventually.

Proof. Assume Eq. (1.1) has a nonoscillatory solution $y(t)$. Then without loss of generality, assume $y(t)>0$ and $y(g(t))>0$ for

$t \in t_0, \infty)$. It follows from Lemmas 2.4 and 2.5 that

$$[L_2(y(t))]' \leq 0 \text{ and } [L_1(y(t))]' > 0,$$

eventually and either $y'(t)$ is eventually positive or $y(t)$ tends to zero eventually. We suppose that

$$[L_2(y(t))]' \leq 0, [L_1(y(t))]' > 0, \quad \text{and } y'(t) > 0,$$

eventually. Then there exists $t_1 \in t_0, \infty)$ such that

$[L_2(y(t))]' \leq 0, [L_1(y(t))]' > 0, \text{and } y'(t) > 0$ for $t \geq t_1$.

Since $y'(t) > 0$ on $[t_1, \infty)$, we have

$$y(t) > y(t_1) =: c > 0.$$

Therefore,

$$q(t)y^\alpha(g(t)) > c^\alpha q(t) \quad \text{for } t \geq t_2, \tag{2.19}$$

where $t_2 = \tau^{-1}(t_1)$. It follows from (1.1) and (2.19) that

$$-[L_2(y(t))]' \geq c^\alpha q(t).$$

Integrating both sides of the last inequality from t_2 to t yields

$$L_2(y(t_2)) > -L_2(y(t)) + L_2(y(t_2)) > c^\alpha \int_{t_2}^{t} q(s)ds,$$

which contradicts (2.18). This completes the proof.

Theorem 2.2 *Let (B_1) and (B_2) hold. Assume for sufficiently large $T_1, T \in t_0, \infty)$,*

$$\int_{T_1}^{\infty} \left(\int_{T}^{g(s)} a_1^{-\frac{1}{\alpha_1}}(v)dv \right)^\alpha q(s)ds = \infty, \tag{2.20}$$

where $g(T_1) \geq T$. Then every solution of Eq. (1.1) *is either oscillatory or tends to nonnegative finite limit eventually.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $y(t)$. Then without loss of generality, assume $y(t) > 0$ and $y(g(t)) > 0$ for $t \in t_0, \infty)$. It follows from Lemmas 2.4 and 2.2 that

$$[L_2(y(t))]' \leq 0 \text{ and } [L_1(y(t))]' > 0,$$

eventually and either $y'(t)$ is eventually positive or $y(t)$ tends to nonnegative finite limit eventually. We suppose that

$$[L_2(y(t))]' \leq 0, [L_1(y(t))]' > 0, \quad \text{and } y'(t) > 0,$$

eventually. Then there exists $t_1 \in t_0, \infty)$ such that

$[L_2(y(t))]' \leq 0, [L_1(y(t))]' > 0, \text{and } y'(t) > 0$ for $t \geq t_1$.

Thus for $t \geq t_1$,

$$y(t) > y(t) - y(t_1)$$

$$= \int_{t_1}^{t} \phi_{\alpha_1}^{-1}[L_1(y(s))] a_1^{-\frac{1}{\alpha_1}}(s)ds$$

$$> \phi_{\alpha_1}^{-1}[L_1(y(t_1))] \int_{t_1}^{t} a_1^{-\frac{1}{\alpha_1}}(s)ds,$$

and so for $t \geq t_2$, where $g(t_2) \geq t_1$,

$$q(t)y^\alpha(g(t)) \geq C\, q(t) \left(\int_{t_1}^{g(t)} a_1^{-\frac{1}{\alpha_1}}(s)ds \right)^\alpha.$$

where $C := \phi_{\alpha_2}[L_1(y(t_1))] > 0$. The rest of the proof is similar to the proof of Theorem 2.1 and hence is omitted.

Theorem 2.3 *Let (B_1) and (B_2) hold. Assume for sufficiently large $T \in t_0, \infty)$, either of the following holds:*

$$\operatorname{Limsup}_{t\to\infty} A_2^\alpha(t) \int_t^\infty \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha q(s)ds > 1, \tag{2.21}$$

$$\operatorname{Limsup}_{t\to\infty} A_2(t) \left(\int_t^\infty \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha q(s)ds \right)^{\frac{1}{\alpha}} > 1. \tag{2.22}$$

Then every solution of Eq. (1.1) *is either oscillatory or tends to nonnegative finite limit eventually.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $y(t)$. Then without loss of generality, assume $y(t) > 0$ and $y(g(t)) > 0$ for $t \in t_0, \infty)$. It follows from Lemmas 2.4 and 2.2 that

$$[L_2(y(t))]' \leq 0 \text{ and } [L_1(y(t))]' > 0,$$

eventually and either $y'(t)$ is eventually positive or $y(t)$ tends to nonnegative finite limit eventually. We suppose that

$$[L_2(y(t))]' \leq 0, [L_1(y(t))]' > 0, \quad \text{and } y'(t) > 0,$$

eventually. Then there exists $t_1 \in t_0, \infty)$ such that

$[L_2(y(t))]' \leq 0, [L_1(y(t))]' > 0, \text{and } y'(t) > 0$ for $t \geq t_1$.

Let $t_2 = \tau^{-1}(t_1)$. By the fact that $y(t)$ is strictly increasing and Eq. (1.1)

we get

$$-[L_2(y(t))]' = q(t)y^\alpha(g(t)) \geq q(t)y^\alpha(\tau(t)) \text{ for } t \geq t_1. \tag{2.23}$$

In view of Lemma 2.3, with $i = 0$, $\frac{y(t)}{A_2(t)}$ is decreasing on $[t_2, \infty)$. Hence

$$y(\tau(t)) \geq \frac{A_2(\tau(t))}{A_2(t)} y(t) \quad \text{for } t \geq t_2. \tag{2.24}$$

Substituting (2.24) into (2.23), we obtain for $t \geq t_2$,

$$-[L_2(y(t))]' \geq q(t) \left[\frac{A_2(\tau(t))}{A_2(t)}\right]^\alpha y^\alpha(t). \tag{2.25}$$

Integrating inequality (2.25) from t to $v \in t, \infty)$, we get

$$-L_2(y(v)) + L_2(y(t)) > \int_t^v q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha y^\alpha(s)ds.$$

Since $L_2(y(v)) > 0$ and by taking limits as $v \to \infty$, we have

$$L_2(y(t)) > \int_t^\infty q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha y^\alpha(s)\, ds.$$

Using the fact that $y(t)$ is strictly increasing, we obtain

$$L_2(y(t)) > \int_t^\infty q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha y^\alpha(s)\, ds$$

$$> y^\alpha(t) \int_t^\infty q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha ds. \tag{2.26}$$

Also, by Lemma 2.3, we have

$$y(t) > A_2(t)\phi_\alpha^{-1}[L_2(y(t))] \quad \text{for } t \geq t_2. \tag{2.27}$$

Assume (2.21) holds. Substituting (2.27) into (2.26), we obtain

$$L_2(y(t)) > L_2(y(t)) A_2^\alpha(t) \int_t^\infty q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha ds.$$

Hence

$$1 > A_2^\alpha(t) \int_t^\infty q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha ds.$$

As a result,

$$\operatorname{Limsup}_{t\to\infty} A_2^\alpha(t) \int_t^\infty q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha ds \leq 1,$$

which contradicts (2.21). Assume (2.22) holds. Substituting (2.26) into (2.27), we obtain

$$y(t) > y(t)\, A_2(t) \left(\int_t^\infty q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha ds \right)^{\frac{1}{\alpha}}.$$

Hence

$$1 > A_2(t) \left(\int_t^\infty q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha ds \right)^{\frac{1}{\alpha}},$$

which implies

$$\operatorname{limsup}_{t\to\infty} A_2(t) \left(\int_t^\infty q(s) \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha ds \right)^{\frac{1}{\alpha}} \leq 1,$$

which contradicts (2.22). This completes the proof.

Theorem 2.4 *Let (B_1) and (B_2) hold. Assume for sufficiently large $T \in t_0, \infty)$, and*

$$\int_T^\infty A_2^\alpha(\tau(s))q(s)ds = \infty. \tag{2.28}$$

Then every solution of Eq. (1.1) *is either oscillatory or tends to* nonnegative finite limit eventually.

Proof. Assume Eq. (1.1) has a nonoscillatory solution $y(t)$. Then without loss of generality, assume $y(t) > 0$ and $y(g(t)) > 0$ for $t \in t_0, \infty)$. It follows from Lemmas 2.4 and 2.2 that

$$[L_2(y(t))]' \leq 0 \text{ and} [L_1(y(t))]' > 0,$$

eventually and either $y'(t)$ is eventually positive or $y(t)$ tends to nonnegative finite limit eventually. We suppose that

$$[L_2(y(t))]' \leq 0, [L_1(y(t))]' > 0, \quad \text{and} y'(t) > 0,$$

eventually. Then there exists $t_1 \in t_0, \infty)$ such that

$$[L_2(y(t))]' \leq 0, [L_1(y(t))]' > 0, \text{and} y'(t) > 0 \text{ for } t \geq t_1.$$

As shown in the proof of Theorem 2.3, we get (2.25) holds for $t \in t_2, \infty)$, where $t_2 = \tau^{-1}(t_1)$. Also in view of Lemma 2.3, we have

$$y(t) > A_2(t)\phi_\alpha^{-1}[L_2(y(t))] \text{ for} t \geq t_2. \qquad (2.29)$$

Substituting (2.29) into (2.25), we obtain for $t \geq t_2$,

$$-[L_2(y(t))]' \geq q(t)A_2^\alpha(\tau(t))L_2(y(t)),$$

or

$$-\frac{[L_2(y(t))]'}{L_2(y(t))} \geq A_2^\alpha(\tau(t))q(t).$$

Integrating the above from t_2 to t, we get

$$-\int_{t_2}^{t} \frac{[L_2(y(s))]'}{L_2(y(s))} ds \geq \int_{t_2}^{t} A_2^\alpha(\tau(s))q(s)ds.$$

It follows that for any $t \in t_2, \infty)$,

$$\int_{t_2}^{t} A_2^\alpha(\tau(s))q(s)ds \leq -\int_{t_2}^{t} \frac{[L_2(y(s))]'}{L_2(y(s))} ds$$

$$= \ln\left(L_2(y(t_2))\right) - \ln\left(L_2(y(t))\right) \leq \ln\left(L_2(y(t_2))\right) < \infty.$$

This contradicts (2.28). This completes the proof.

Corollary 2.1 *Let (B_1), (B_2) and (B_3) hold. If either (2.20), (2.21), (2.22) or (2.28) satifies, then every solution of Eq. (1.1) is either oscillatory or tends to zero eventually.*

Example 2.1 *Consider the third order nonlinear differential equation*

$$\left\{t^{\alpha_2-\gamma}\phi_{\alpha_2}([t^{\alpha_1-\gamma}\phi_{\alpha_1}(y'(t))]')\right\}' + \frac{1}{t^{1-\gamma}}\phi_\alpha\left(y(g(t))\right) = 0 \text{ for } t \in [t_0, \infty). \qquad (2.30)$$

where $\alpha_i > 0, i = 1,2$ and $\gamma \geq 0$ are constants. We have

$$a_1(t) = t^{\alpha_1-\gamma}, a_2(t) = t^{\alpha_2-\gamma}, q(t) = \frac{1}{t^{1-\gamma}}, \text{for } t \in [t_0, \infty).$$

Since

$$\int_{t_0}^{\infty} a_1^{\frac{-1}{\alpha_1}}(s)ds = \int_{t_0}^{\infty} \frac{ds}{s^{1-\frac{\gamma}{\alpha_1}}} = \infty,$$

$$\int_{t_0}^{\infty} a_2^{\frac{-1}{\alpha_2}}(s)ds = \int_{t_0}^{\infty} \frac{ds}{s^{1-\frac{\gamma}{\alpha_2}}} = \infty, \quad \text{and}$$

$$\int_{T}^{\infty} q(s)ds = \int_{T}^{\infty} \frac{ds}{s^{1-\gamma}} = \infty.$$

Then, by Theorem 2.1, every solution of (2.30) is either oscillatory or tends to zero eventually.

Example 2.2 *Consider the third order nonlinear differential equation (1.1) when* $g(t) \geq t$ *and*

$$q(t) := \frac{\gamma}{A_2^{\alpha+1}(t)}\left(\frac{A_1(t)}{a_1(t)}\right)^{\frac{1}{\alpha_1}},$$

where $\gamma > 0$ is a constant with $\gamma > \alpha$. Choose a_i satisfying (B_1) and (B_2). Since

$$\left(-\frac{1}{A_2^\alpha(t)}\right)' = \frac{\alpha A_2'(t)}{A_2^{\alpha+1}(t)} = \frac{\alpha}{A_2^{\alpha+1}(t)}\left(\frac{A_1(t)}{a_1(t)}\right)^{\frac{1}{\alpha_1}}.$$

Therefore

$$\limsup_{t\to\infty} A_2^\alpha(t) \int_t^\infty \left[\frac{A_2(\tau(s))}{A_2(s)}\right]^\alpha q(s)\, ds$$

$$= \frac{\gamma}{\alpha} \limsup_{t\to\infty} A_2^\alpha(t) \int_t^\infty \frac{\alpha}{A_2^{\alpha+1}(s)}\left(\frac{A_1(s)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} ds$$

$$= \frac{\gamma}{\alpha} \limsup_{t\to\infty} A_2^\alpha(t) \int_t^\infty \left(-\frac{1}{A_2^\alpha(t)}\right)' ds = \frac{\gamma}{\alpha} > 1.$$

Then, by Theorem 2.3, *every solution of* Eq. (1.1) is either oscillatory or tends to nonnegative finite limit eventually.

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