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Mathematics

LAPLACE TRANSFORMS AND ITS APPLICATIONS TO DIFFERENTIAL EQUATION

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ABSTRACT The Laplace transform provides a simple and effective process for process for solution to many problems arising in various fields of science. Laplace transform is particularly use to solve linear ordinary differential equations. The Laplace transform helps solve complex problems with a very simple approach to solving ordinary differential equations.

KEYWORDS : Laplace transform, Properties, Inverse Laplace transformation, Differential equations.

INTRODUCTION

The Laplace transform is a special type of integral transform. Consider a function f (t), its corresponding Laplace transform will be denoted as Lf(t), Where L is Laplace transform operator.[4]

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

provided that the integral exists, where s is a parameter which may be real or complex number.[1]

2 Properties Of Laplace Transform

Here are some important properties of Laplace transform[5]

2.1 Linearity Property

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

where a and b are constant

2.2 First Shifting Theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{at}f(t)\} = F(s-a)$

2.3 Laplace Transform Of Nth Derivative

If $L\{f(t)\} = F(s)$, then $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots\cdots - f^{n-1}(0)$

2.4 Laplace Transform Of Integrals

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$

2.5 Multiplication By t^n

If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$

3 Application To Differential Equation

Laplace transform method of solving differential equations in particular solution without the need to first find the general solution and then invalidate arbitrary constants[6].

This method is generally shorter than our previous methods and is particularly use to solve linear differential equations with constant coefficients.[2] Consider a linear differential equation of the form

$$L\{ a_n \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)\} = L\{f(t)\}$$

3.1 Consider The Differential Equation

$$\frac{dy(t)}{dt} + 2y(t) = 3e^{-t}, \quad y(0) = 1$$

Take the Laplace transform of both sides.

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} + \mathcal{L}\{2y(t)\} = \mathcal{L}\{3e^{-t}\}$$

Using the properties of Laplace transforms

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = sY(s) - y(0)$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

Substituting these into the equation

$$sY(s) - 1 + 2Y(s) = \frac{3}{s+1}$$

or,

$$(s+2)Y(s) = \frac{3}{s+1} + 1$$

or,

$$(s+2)Y(s) = \frac{3}{s+1} + \frac{s+1}{s+1}$$

or,

$$(s+2)Y(s) = \frac{s+4}{s+1}$$

or,

$$Y(s) = \frac{s+4}{(s+1)(s+2)}$$

Using partial fractions

$$\frac{s+4}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

Multiplying both sides by (s+1)(s+2)

$$s+4 = A(s+2) + B(s+1)$$

or,

$$s+4 = (A+B)s + (2A+B)$$

Equating the coefficients

$$A+B=1, \quad 2A+B=4$$

Solving, we get A=3 and, B=-2, so:

$$Y(s) = \frac{3}{s+1} - \frac{2}{s+2}$$

Taking the inverse Laplace transform

$$y(t) = 3e^{-t} - 2e^{-2t}$$

The required solution is

$$y(t) = 3e^{-t} - 2e^{-2t}$$

3.2 To Solve The Differential Equation

$$y''' + 2y'' - y' - 2y = 0$$

with initial conditions y (0)=0, y'(0)=0, and y''(0)=6 using the Laplace transform,

We apply the Laplace transform to each term, using the properties of the Laplace transform of derivatives:

$$\mathcal{L}(y^{(n)}(t)) = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$$

Using this for each term:

$$\mathcal{L}(y''') = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

With the initial conditions y(0)=0, y'(0)=0, and, y''(0)=6 this becomes:

$$\mathcal{L}(y''') = s^3Y(s) - 6$$

$$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

Since y(0)=0 and, y'(0)=0 this simplifies to:

$$\mathcal{L}(y'') = s^2Y(s)$$

$$\mathcal{L}(y') = sY(s) - y(0)$$

With y(0)=0, we get:

$$\mathcal{L}(y') = sY(s)$$

$$\mathcal{L}(y) = Y(s)$$

Putting these Laplace transforms into the differential equation:

$$s^3Y(s) - 6 + 2s^2Y(s) - sY(s) - 2Y(s) = 0$$

or,

$$s^3Y(s) - 6 + 2s^2Y(s) - sY(s) - 2Y(s) = 0$$

or,

$$Y(s)(s^3 + 2s^2 - s - 2) = 6$$

or,

$$Y(s) = \frac{6}{s^3+2s^2-s-2}$$

Now, we decompose $\frac{6}{(s+1)(s+2)(s-1)}$ into partial fractions:

$$\frac{6}{(s+1)(s+2)(s-1)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-1}$$

Multiply both sides by (s+1)(s+2)(s+3):

$$6 = A(s+2)(s-1) + B(s+1)(s-1) + C(s+1)(s+2)$$

or,

$$6 = A(s^2 + s - 2) + B(s^2 - 1) + C(s^2 + 3s + 2)$$

or,

$$6 = (A + B + C)s^2 + (A + 3C)s + (-2A - B + 2C)$$

we equate the coefficients of s^2, s, and the constant term:

$$A + B + C = 0$$
$$A + 3C = 0$$
$$-2A - B + 2C = 6$$

From A+3C=0, we have A= -3C. Substituting into: A+B+C=0:

$$-3C + B + C = 0 \quad \Rightarrow \quad B = 2C$$

Substitute A=-3C and B=2C into -2A -B +2C = 6.

$$-2(-3C) - 2C + 2C = 6 \quad \Rightarrow \quad 6C = 6 \quad \Rightarrow \quad C = 1$$

Thus, A=-3 and B=2.

The partial fraction decomposition is:

$$\frac{6}{(s+1)(s+2)(s-1)} = \frac{-3}{s+1} + \frac{2}{s+2} + \frac{1}{s-1}$$

In each term we take the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{-3}{s+1}\right) = -3e^{-t}$$

$$\mathcal{L}^{-1}\left(\frac{2}{s+2}\right) = 2e^{-2t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^{t}$$

Thus, the required solution is:

$$y(t) = -3e^{-t} + 2e^{-2t} + e^{t}$$

4 CONCLUSION

This paper consisted of a brief overview of Laplace Transform. Laplace transform is a very effective mathematical tool to simplify very complex problems. In these days there is tremendous use of Laplace transform to find the solution of different problems.

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