RESEARCH PAPER | Engineering | Volume : 3 | Issue : 4 | April 2013 | ISSN - 2249-555X

A Survey on Mathematical Functionalities of Segmenting and Filtering Digital Images

KEYWORDS: Digital image processing, Discrete Fourier Transform, Convolution, Matrices, MSE, PSNR.

Rupini. T	Sarjana Devi. S	C. Santhosh Kumar
Bachelor of Engineering, Dept of CSE, Narasu's Sarathy Institute of Technology, Salem	Bachelor of Engineering, Dept of CSE, Narasu's Sarathy Institute of Technology, Salem	Asst.Prof, Dept of CSE, Narasu's Sarathy Institute of Technology, Salem

ABSTRACT *The image based computing is enhancing in the world through their intrusion in real life. The mathematical ideas behind that will bring some new knowledge and creativeness. An image is considered to be a function of two real variables with amplitude at the real coordinate position (x,y). The amplitudes of a given image will almost be either real numbers or integer numbers. The mathematical tools includes fourier transform, convolution and matrices. The Fourier Transform analyze boundary value problems including both ordinary and partial differentiations written in form of finite sum using convolution. Matrix can be predicted in a image using its particular pixel value . Peak Signal to Noise gives approximation to human perception and reconstruction of higher quality images. This paper presents an overview of showing how the various mathematical formulations through their tools are used to process digital images.*

1. Introduction

A digital image a[m,n] described in a 2D discrete space is derived from an analog image a(x,y) in a 2D continuous space through a sampling process that is frequently referred to as digitization. [10]The 2D continuous image a(x,y) is divided into N rows and M columns. [4]The intersection of a row and a column is termed a pixel. The value assigned to the integer coordinates [m,n] with {m=0,1,2,…,M–1} and {n=0,1,2,…,N–1} is a[m,n]. In fact, in most cases a(x,y)—which we might consider to be the physical signal that impinges on the face of a 2D sensor—is actually a function of many variables including depth (z), color (l), and time (t).

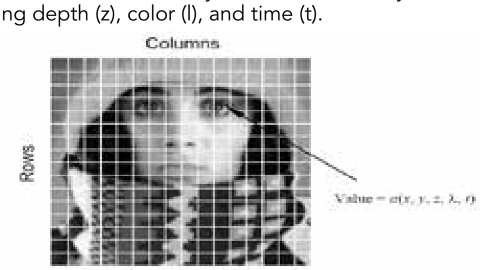

Fig:1 Digitization of a continuous image.

The pixel at coordinates [m=10, n=3] has the integer brightness value 110.The image shown in Fig:1 has been divided into N = 16 rows and M = 16 columns. [6]The value assigned to every pixel is the average brightness in the pixel rounded to the nearest integer value.

2. Common Values

There are standard values for the various parameters encountered in digital image processing. These values can be caused by video standards, by algorithmic requirements, or by the desire to keep digital circuitry simple. Table 1 gives some commonly encountered values.

Parameter	*Symbol*	*Typical values*
Rows	N	256,512,525,625,1024,1035
Columns	M	256,512,768,1024,1320
Gray Levels	L	2,64,256,1024,4096,16384

Table 1: Common values of digital image parameters

Quite frequently we see cases of M=N=2K where {K = 8,9,10}. [5]This can be motivated by digital circuitry or by the use of certain algorithms such as the (fast)Fourier transform. The number of distinct gray levels is usually a power of 2, that is, L=2B where B is the number of bits in the binary representation of the brightness levels. When B>1we speak of a gray-level image; when B=1 we speak of a binary image. In a binary image there are just two gray levels which can be referred to, for example, as "black" and "white" or "0" and "1".

Certain tools are central to the processing of digital images. These include mathematical tools such as matrices, convolution and Fourier analysis , and manipulative tools such as chain codes and run codes. We will present these tools without any specific motivation.

3. Fourier Transforms

Fourier Transform is an integral transform of one function into another. The Fourier transform provides information about the global frequency-domain characteristics of an image. The Fourier description can be computed using discrete techniques, which are natural for digital images. Fourier series are designed to solve boundary value problems on bounded intervals. [1]The extension of the Fourier calculus to the entire real line leads naturally to the Fourier transform, a powerful mathematical tool for the analysis of non-periodic functions. The inverse Fourier transform reconstructs the original function from its transformed frequency components.

3.1. Discrete Fourier Transform In Image Processing

The Fourier Transform is an important image processing tool which is used to decompose an image into its sine and cosine components. The output of the transformation represents the image in the Fourier or frequency domain, while the input image is the spatial domain equivalent. In the Fourier domain image, each point represents a particular frequency contained in the spatial domain image.

In image processing, the Fourier Transform is used in a wide range of applications, such as image analysis, image filtering, image reconstruction and image compression. The DFT is the sampled Fourier Transform. That is why DFT does have all the frequences which form the image, but only a set of samples which is large enough to fully describe the spatial domain image. The number of frequencies corresponds to the number of pixels in the spatial domain image, i.e. the

Volume : 3 | Issue : 4 | April 2013 | ISSN - 2249-555X

image in the spatial and Fourier domain are of the same size. [2]For a square image of size N×N, the two-dimensional DFT is given by:

$$F(k,l) = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} f(i,j)\, e^{-i2\pi\left(\frac{ki}{N}+\frac{lj}{N}\right)}$$

f(i,j) is the image in the spatial domain and the exponential term is the basis function corresponding to each point F(k,l) in the Fourier space. The equation can be interpreted as: the value of each point F(k,l) is obtained by multiplying the spatial image with the corresponding base function and summing the result. The basis functions are sine and cosine waves with increasing frequencies, i.e. F(0,0) represents the DC-component of the image which corresponds to the average brightness and F(N-1,N-1) represents the highest frequency.

In a similar way, the Fourier image can be re-transformed to the spatial domain. The inverse Fourier transform is given by:

$$f(i,j) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F(k,l)\, e^{i2\pi\left(\frac{ki}{N}+\frac{lj}{N}\right)}$$

To obtain the result for the previous equations, a double sum has to be calculated for each image point. However, because the Fourier Transform is separable, it can be written as:

$$f(k,l) = \frac{1}{N}\sum_{j=0}^{N-1} P(k,j)\, e^{-i2\pi\frac{lj}{N}}$$

Where,

$$P(k,j) = \frac{1}{N}\sum_{i=0}^{N-1} f(i,j)\, e^{-i2\pi\frac{ki}{N}}$$

Using those last two formulas, the spatial domain image is first transformed into an intermediate image using N one-dimensional Fourier Transforms. This intermediate image is then transformed into the final image, again using N one-dimensional Fourier Transforms.

Expressing the two-dimensional Fourier Transform in terms of a series of 2N one-dimensional transforms decreases the number of required computations. The ordinary one-dimensional DFT still has complexity which can be reduced with the use of Fast Fourier Transform (FFT) to compute the onedimensional DFTs.[3] It is a significant improvement, in particular for large images. There are various forms of the FFT and most of them restrict the size of the input image that may be transformed, often to where n is an integer. The Fourier Transform produces a complex number valued output image which can be displayed with two images, either with the real and imaginary part or with magnitude and phase.

In image processing, often only the magnitude of the Fourier Transform is displayed, as it contains most of the information of the geometric structure of the spatial domain image. The Fourier image can also be re-transformed into the correct spatial domain after some processing in the frequency domain...(both magnitude and phase of the imgae must be preserved for this). The Fourier domain image has a much greater range than the image in the spatial domain. Hence, to be sufficiently accurate, its values are usually calculated and stored in float values.

4. Convolution

There are several possible notations to indicate the convolution of two (multidimensional) signals to produce an output signal. The most common are:

$$c = a \otimes b = a * b$$

We shall use the above form, with the following formal definitions.

$$c(x,y) = a(x,y) \otimes b(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a(\chi,\zeta)\, b(x-\chi, y-\zeta)\, d\chi\, d\zeta$$

In 2D discrete space:

$$c[m,n] = a[m,n] \otimes b[m,n] = \sum_{j=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} a[j,k]\, b[m-j, n-k]$$

4.1. Properties Of Convolution

There are a number of important mathematical properties associated with convolution.

- Convolution is commutative.
  $c=a\otimes b=b\otimes a$
- Convolution is associative.
  $c=a\otimes(b\otimes d)=(a\otimes b)\otimes d=a\otimes b\otimes d$
- Convolution is distributive.
  $c=a\otimes(b+d)=(a\otimes b)+(a\otimes d)$

where a, b, c, and d are all images, either continuous or discrete

5.Matrix

A matrix is a grid, with each location in the grid containing some information. For example, a chess board is a matrix in which every square contains a specific item of information: a particular chess piece, or the lack of a chess piece.

5.1 Linear equation:

A matrix can be derived from an linear equation. A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and (the first power of) a single variable.

5.2 Forms for 2D linear equations:

Linear equations can be rewritten using the laws of elementary algebra into several different forms. These equations are often referred to as the "equations of the straight line." In what follows, x, y, t, and θ are variables; other letters represent constants (fixed numbers).

5.2.1. General form

Ax+By+C=0, where A and B are not both equal to zero. The equation is usually written so that A ≥ 0, by convention. The graph of the equation is a straight line, and every straight line can be represented by an equation in the above form. If A is nonzero, then the x-intercept, that is, the x-coordinate of the point where the graph crosses the x-axis (where, y is zero), is −C/A. If B is nonzero, then the y-intercept, that is the y-coordinate of the point where the graph crosses the y-axis (where x is zero), is −C/B, and the slope of the line is −A/B.

5.2.2. Standard form

Ax+By+C=0, where A and B are not both equal to zero, A, B, and C are coprime integers, and A is nonnegative (if zero, B must be positive). The standard form can be converted to the general form, but not always to all the other forms if A or B is zero. It is worth noting that, while the term occurs frequently in school-level US algebra textbooks, most lines cannot be described by such equations. For instance, the line $x + y = \sqrt{2}$ cannot be described by a linear equation with integer coefficients since $\sqrt{2}$ is irrational.

5.2.3. Matrix form

Using the order of the standard form

Ax+By=C, but without the restriction of coprime integer coefficients one can rewrite the equation in matrix form:

$$\begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C \end{pmatrix}$$

Further, this representation extends to systems of linear equations.

$A_1x+B_1y=C_1$,
$A_2x+B_2y=C_2$,

Volume : 3 | Issue : 4 | April 2013 | ISSN - 2249-555X

Becomes

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Since this extends easily to higher dimensions, it is a usual method in linear algebra, and in computer programming. In particular, there are named methods for solving simultaneous linear equations like Gauss-Jordan which can be expressed as matrix elementary row operations.

5.3. Linear Transformation matrices

[7]In two-dimensional space R^2 linear maps are described by 2 × 2 real matrices. These are some examples:

- rotation by 90 degrees counterclockwise:
  $A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- rotation by θ degrees counterclockwise:
  $A=\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
- reflection against the x axis:
  $A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- reflection against the y axis:
  $A=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- scaling by 2 in all directions:
  $A=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
- horizontal shear mapping:
  $A=\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$
- squeeze mapping:
  $A=\begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}$
- projection onto the y axis:
  $A=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

A single linear map may be represented by many matrices. This is because the values of the elements of the matrix depend on the bases that are chosen. [8]This can be applied for geometric transformations, such as those performed in computer graphics where the translation, rotation and scaling of 2D or 3D objects is performed by the use of a transformation matrix.

6. PSNR and MSE

The phrase peak Signal-to-Noise Ratio, often abbreviated PSNR, is the ratio between the maximum possible power of a signal and the power of corrupting noise that affects the fidelity of its representation. Because many signals have a very wide dynamic range, PSNR is usually expressed in terms of the logarithmic decibel scale. It is most easily defined via the mean squared error (MSE). Given a noise-free m×n monochrome image I and its noisy approximation K, MSE is defined as:

$$MSE=\frac{1}{mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[I(i,j)-K(i,j)]^2$$

$$PSNR = 10\log_{10}\left[\frac{255^2}{MSE}\right]$$

The PSNR is most commonly used as a measure of quality of reconstruction of lossy compression codecs (e.g., for image compression). The signal in this case is the original data, and the noise is the error introduced by compression. [9]When comparing compression codecs it is used as an approximation to human perception of reconstruction quality, therefore in some cases one reconstruction may appear to be closer to the original than another, even though it has a lower PSNR (a higher PSNR would normally indicate that the reconstruction is of higher quality). One has to be extremely careful with the range of validity of this metric; it is only conclusively valid when it is used to compare results from the same codec (or codec type) and same content.

7. Conclusion

In this paper a Fourier Transform is used to decompose an image into its sine and cosine components. A particularly important consequence is that it effectively transforms differential equations into algebraic equations, and thereby facilitates their solution by elementary algebra .Solving the resulting algebraic equation will produce a formula for the Fourier transform of the desired solution, which can then be immediately reconstructed via the inverse Fourier transform. Here non-periodic functions of an image are analysed by using a fourier transform and the resulted differential or integral equation is converted into algebraic equation by which a matrix can be constructed. The required transformations can be applied to the constructed matrix and a new matrix is derived which then be converted into an algebraic equation and an inverse Fourier Transform is applied to reconstruct the original image. The PSNR value is used to give the quality of an image. Higher the PSNR value the quality of an image will be high. It is used in the image reconstruction process. It has an applications in Image reconstruction, Image analysis, Image filtering, Image compression.

REFERENCE [1] Joonas Konttinen, Przemysław Pyka, Mika Kangas, "Fourier Transform in Image Processing".Lappeenranta University of Technology. | [2] Harvey Rhody, C. F. (october 14,2005). "Lecture 11: Fourier Transform Image". | [3] Peter J. Olver," Chapter 8: Fourier Transforms". | [4] Ian T. Young, Jan J. Gerbrands, Lucas J. van Vliet "Fundamentals of Image Processing" , ISBN 90– 75691–01–7. CIP-DATA KONINKLIJKE BIBLIOTHEEK, DenHaang. | [5] Jitter,"tutorials". | [6] Barnett, R.A.; Ziegler, M.R.; Byleen, K.E. (2008), "College Mathematics for Business" | [7] wikipedia,"linear map, linear transformation,matrices" | [8] Halmos, Paul R. (1974), Finite-dimensional vector spaces, New York | [9]L. Ambrosio and V.M. Tortorelli, Communications on Pure and Applied Mathematics, Vol. XLIII, pp. | 999-1036, 1990. | [10] R.C.Gonzalez and R.E. Woods , Digital Image Processing. Addison Wesley, 2002.