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Numerical Solution of Fuzzy Initial Value Problems by Fourth Order Runge-Kutta Method Based on Contraharmonic Mean

KEYWORDS Numerical solution, Fuzzy differential equation, Runge-Kutta method, Contraharmonic mean, Trapezoidal fuzzy number.

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ABSTRACT *In this paper, a numerical algorithm for solving fuzzy initial value problem based on Seikkala's derivative of fuzzy process by the fourth order Runge-Kutta methods based on Contraharmonic Mean(RKCoM) is proposed in detail. The algorithm is illustrated by solving a linear and non linear fuzzy initial value problems(IVPs) using trapezoidal fuzzy number. It is also shown that in the proposed method convergence order is O(h4). The results show that the proposed methods suits very well to solve linear and non linear fuzzy initial value problems.*

1. Introduction

Fuzzy set theory is a tool that makes possible to describe vague and uncertain notions. Fuzzy Differential Equation (FDE) models have wide range of applications in many branches of engineering and in the field of medicine. The concept of a fuzzy derivative was first introduced by Chang and Zadeh [8], later Dubois and Prade [9] defined the fuzzy derivative by us in Zadeh's extension principle and then followed by Puri and Ralescu [23]. Fuzzy differential equations have been suggested as a way of modelling uncertain and incompletely specified systems and were studied by many researchers [11, 12, 15]. The existence of solutions of fuzzy differential equations has been studied by several authors [3, 4]. It is difficult to obtain exact solution for fuzzy differential equations and hence several numerical methods where proposed [17]. Abbasbandy and Allahviranloo [2] developed numerical algorithms for solving fuzzy differential equations based on Seikkala's derivative of fuzzy process [25]. Runge-Kutta method for fuzzy differential equation has been studied by many authors [1, 22]. Kanagarajan and Sampath [13, 14] developed a numerical algorithm for solving fuzzy differential equations by using Runge-Kutta method and Runge-Kutta Nystrom method of order three. Nirmala and Chenthur Pandian [21] studied numerical solution of fuzzy differential equation by fourth order Runge-Kutta method with higher order derivative approximations. Evans and Yaacub [10] have introduced the fourth order Runge-Kutta method based on Centroidal Mean (RKCeM) formula for first order IVPs. Murugesan et al. [19] compared fourth order RK methods based on variety of means and concluded that RKCeM works very well to solve system of IVPs and they also developed [20] a new embedded RK method based on AM and CeM. The applicability of the RKCeM : Division by zero, Error in RKCeM formulae, and Stability analysis are discussed by Murugesan et al. [18].

In this paper, the fourth order RKCoM is applied to solve fuzzy initial value problems and established that the approximate solution of the proposed fourth order RKCoM almost coincides with the exact solution.

The structure of the paper is organized as follows: In Section 2, some necessary notations and definitions of fuzzy set theory, fuzzy differential equations, fourth order Runge-Kutta formula based on Contraharmonic Mean to solve IVPs are given. In Section 3, Fuzzy initial value problem is defined. In section 4, solving numerically the fuzzy initial value problems by the fourth order Runge-Kutta method based on the proposed method is discussed and gives the convergence result. The proposed algorithm is illustrated by an example in section 5 and the conclusion is in section 6.

2. Preliminaries

Definition 2.1. A fuzzy number is a fuzzy set u : $\mathbb{R} \to [0, 1]$ which satisfies

1. u is upper semicontinuous.
2. u(x) = 0 outside some interval [c, d],
3. there are real numbers a, b for which $c \le a \le b \le d$ such that

3.1. u(x) is monotonic increasing on [c, a],
3.2. u(x) is monotonic decreasing on [b, d],and
3.3. u(x) = 1, $a \le x \le b$.

Definition 2.2. A fuzzy number u in parametric form is a pair $u = (\underline{u}(r), \overline{u}(r)),\ r \in [0, 1],$ which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous monotonic increasing function over [0, 1],
2. $\overline{u}(r)$ is a bounded left continuous monotonic decreasing function over [0, 1], and
3. $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \le r \le 1.$

Definition 2.3.
A trapezoidal fuzzy number u, is defined by four real numbers k < l < m < n where the base of the trapezoidal is the interval [k, n] and its vertices at x=l, x=m. Trapezoidal fuzzy number will be written as u = (k, l, m, n). The membership function for the trapezoidal fuzzy number u = (k, l, m, n) is defined as the following :

$$u(x) = \begin{cases} \dfrac{x-k}{l-k}, & k \le x \le l \\ 1, & l \le x \le m \\ \dfrac{x-n}{m-n}, & m \le x \le n \end{cases}$$

and one can have :

(1) u > 0 if k > 0 (2) u > 0 if l > 0;(3) u > 0 if m > 0; and (4) u > 0 if n > 0.

Let E be the set of all upper semi continuous normal convex fuzzy numbers with bounded r−level intervals. It means that is $v \in E$ then r-level set

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$$[v]_r = \{s \setminus v(s) \geq r\},\ 0 < r \leq 1,$$

is a closed bounded interval which is denoted by $[v]_r = [v_1(r),\ v_2(r)].$

Lemma 2.1. Let v, w ∈ E and s a scalar, then for r ∈ (0, 1]

$$[v+w]_r = [v_1(r)+w_1(r),\ v_2(r)+w_2(r)],$$
$$[v-w]_r = [v_1(r)-w_1(r),\ v_2(r)-w_2(r)],$$
$$[v\cdot w]_r = [\min\{v_1(r)\cdot w_1(r), v_1(r)\cdot w_2(r),\ v_2(r)\cdot w_1(r), v_2(r)\cdot w_2(r)\},$$
$$\max\{v_1(r)\cdot w_1(r), v_1(r)\cdot w_2(r),\ v_2(r)\cdot w_1(r), v_2(r)\cdot w_2(r)\}].$$
$$[sv]_r = s[v]r.$$

Definition 2.4. For arbitrary fuzzy numbers $u = (\underline{u}(r),\ \overline{u}(r)),$ and $v = (\underline{v}(r),\ \overline{v}(r)),$ the Quantity (2.1)

$$D(u,\ v) = \sup 0 \leq r \leq 1 \left\{ \max\left[|\underline{u}(r) - \underline{v}(r)|,\ |\overline{u}(r) - \overline{v}(r)| \right] \right\},$$

is the distance between u and v.

The function D (u, v) is a metric on E^1. This metric function is equivalent to the one used by Puri and Ralescu [23] and Kaleva [11].

Definition 2.5. A function $f : \mathbb{R} \to E^1$ is called a fuzzy function. If for arbitrary fixed $t_0 \in \mathbb{R}$ and ∈ >0 δ>0 such that

$$|t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \in \qquad (2.2)$$

exists, f is said to be continuous.

Suppose that $y : I \to E^1$ is a fuzzy function. The parametric form of y (t) is represented by $[y(t)]_r = [y_1(t,\ r),\ y_2(t,\ r)],$ t∈ I, r ∈ (0, 1], (2.3)

where I is a real interval. The Seikkala [25] derivative $y'(t)$ of a fuzzy function y (t) is defined by

$[y'(t)]_r = [y_1'(t,\ r),\ y_2'(t,\ r)]$, t∈ I, r (0, 1], (2.4)

provided that this equation defines a fuzzy number.

2.1. The fourth order Runge-Kutta method based on Contraharmonic Mean

Consider the initial value problem

$$\frac{dy}{dt} = f(t, y(t)),\ a \leq t \leq b \qquad (2.5)$$
$$y(a) = \alpha$$

The basis of all Runge-Kutta method is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=1}^{m} w_i k_i \qquad (2.6)$$

where for i = 1, 2, …, m, wi's are constants and

$$k_i = h.f\left(t_n + c_i h,\ y_n + h\sum_{i=1}^{i-1} a_{ij} k_j\right) \qquad (2.7)$$

Equations (2.7) is to be exact for powers of h through h^m, because it is to be coincident with Taylor series of order m. Therefore, the truncation error Tm, can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}) \qquad (2.8)$$

For $y' = f(t, y),$ the fourth order Runge-Kutta method using Contraharmonic Mean can be written in the form

$$y_{n+1} = y_n + \frac{h}{3}\left[\sum_{i=1}^{3} Means\right] \qquad (2.9)$$

where means includes Contraharmonic Mean(CoM), which involves $k_i,\ 1 \leq i \leq 4$,

where,

$$k_1 = f(t_n,\ y_n)$$
$$k_2 = f(t_n + a_1\ h,\ y_n + a_1\ h\ k_1)$$
$$k_3 = f(t_n + (a_2 + a_3)\ h,\ y_n + a_2\ h\ k_1 + a_3\ h\ k_2)$$
$$k_4 = f(t_n + (a_4 + a_5 + a_6)\ h,\ y_n + (a_4\ h\ k_1 + a_5\ h\ k_2 + a_6\ h\ k_3)) \qquad (2.10)$$

where the parameters for

Contraharmonic Mean :

$$a_1 = \tfrac{1}{2},\ a_2 = \tfrac{1}{8},\ a_3 = \tfrac{3}{8},\ a_4 = \tfrac{1}{4},\ a_5 = \tfrac{-3}{4},\ a_6 = \tfrac{3}{2}. \qquad (2.11)$$

The fourth order formulae based on Runge-Kutta scheme using Contraharmonic Mean is as follows:

Contraharmonic Mean : $y_{n+1} = y_n + \frac{h}{3}\left(\frac{k_1^2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_4^2}{k_3 + k_4}\right)$ (2.12)

$a = t_0 < t_1 < ... < t_N = b$ and $h = \frac{(b-a)}{N} = t_{i+1} - t_i$ (2.13)

The local truncation error (LTE) of the methods are given by the following:

RKCoM:

$$LTE_{CoM} = \frac{h^5}{23040}\left[-378ff_y^4 - 8f^4 f_{yyyy} + 4f^3 f_y f_{yyy} - 648f^3 f_{yy}^2 - 303f^2 f_y^2 f_{yy}\right] + O(h^6) \qquad (2.14)$$

Theorem 2.1. Let $f(t,\ y)$ belongs to C4[a, b] and let it's partial derivatives are bounded and assume that there exists L, M, positive constants such that

$$|f(t,\ y)| < M \quad \left|\frac{\partial^{i+j} f}{\partial t^i \partial y^j}\right| < \frac{L^{i+j}}{M^{j-1}},\ i + j < m.$$

then in terms of the error bound due to Lotkin (see Lambert [25], we have a strict upper bound (with respect to y only) as in the fourth order Runge-Kutta method based on

Contraharmonic Mean : (2.15)

$$|y(t_{i+1}) - y_{i+1}| = |LTE_{CoM}| \leq \frac{1333}{23040} h^5 M L^4 + O(h^6)$$

3. Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$y'(t) = f(t, y(t)\ ;\ 0 \leq t \leq T$$
$$y(0) = y_0, \qquad (3.1)$$

with the grid points (3.2)

$$0 \leq t_1 \leq t_2 \leq ... \leq t_N = T\ and\ h = \frac{(b-a)}{N} = t_{i+1} - t_i$$

where f is a continuous mapping from R+ × R into R and y0 ∈ E with r-level sets

$$[y_0]_r = [y_1(0;\ r),\ y_2(0;\ r)],\ r \in (0,\ 1],$$

The extension principle of Zadeh leads to the following definition of f (t, y) when y = y(t) is a fuzzy number

$$f(t,\ y)(s) = \sup\{y(\tau) \setminus s = f(t,\ \tau)\},\ s \in R$$

It follows that

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$$[f(t, y)]_r = [f_1(t, y; r), f_2(t, y; r)], \ r \in (0, 1],$$

where

$$f_1(t, y; r) = \min\{f(t, u) \backslash u \in [y_1(r), y_2(r)]\}$$
$$f_2(t, y; r) = \max\{f(t, u) \backslash u \in [y_1(r), y_2(r)]\} \quad (3.3)$$

Theorem 3.1. Let f satisfy

$$\left|f(t, v) - f(t, \bar{v})\right| \le g(t, \left|v - \bar{v}\right|), \ t \ge 0, \ v, \ \bar{v} \in R.$$

where g : $R_+ \times R_+ \to R_+$ is a continuous mapping such that r $\to$ g(t, r) is non decreasing and the initial value problem

$$u'(t) = g(t, u(t)), \ u(0) = u_0. \quad (3.4)$$

has a solution on R+ for $u_0 > 0$ and that u(t) $\equiv$ 0 is the only solution of (3.4) for $u_0 = 0$. Then the fuzzy initial value problem (3.1) has a unique solution.

Proof: see [25].

4. The fourth order RK methods for solving Fuzzy Initial Value Problems

4.1. The Fourth Order Runge-Kutta method based on Contraharmonic Mean

We consider fuzzy initial value problem (3.1) with the grid points (3.2)

Let the exact solution [Y (t)]r = [Y1(t; r), Y2(t; r)] is approximated by some [y(t)]r = [y1(t; r), y2(t; r)].

From (2.6), (2.7) we define

$$y_1(t_{n+1}; r) - y_1(t_n; r) = \sum_{i=1}^{4} w_i \ k_{i,1}(t_n, y(t_n; r)), \quad (4.1)$$

$$y_2(t_{n+1}; r) - y_2(t_n; r) = \sum_{i=1}^{4} w_i \ k_{i,2}(t_n, y(t_n; r)),$$

where the wi's are constants and

$$[k_i(t, y(t; r))]_r = [k_{i,1}(t, y(t; r), k_{i,2}(t, y(t; r))], \ i = 1, 2, 3, 4$$

$$k_{i,1}(t_n, y(t_n; r)) = h f(t_n + c_i h, y_1(t_n) + \sum_{j=1}^{i-1} a_{ij} \ k_{j,1}(t_n, y(t_n; r))), \quad (4.2)$$

$$k_{i,2}(t_n, y(t_n; r)) = h f(t_n + c_i h, y_2(t_n) + \sum_{j=1}^{i-1} a_{ij} \ k_{j,2}(t_n, y(t_n; r))),$$

and

$$k_{1,1}(t, y(t; r)) = \min\{h f(t, u) \backslash u \in [y_1(t; r), y_2(t; r)]\}$$

$$k_{1,2}(t, y(t; r)) = \max\{h f(t, u) \backslash u \in [y_1(t; r), y_2(t; r)]\}$$

$$k_{2,1}(t, y(t; r)) = \min\{h f(t + \tfrac{h}{2}, u) \backslash u \in [z_{1,1}(t, y(t; r)), z_{1,2}(t, y(t; r))]\}$$

$$k_{2,2}(t, y(t; r)) = \max\{h f(t + \tfrac{h}{2}, u) \backslash u \in [z_{1,1}(t, y(t; r)), z_{1,2}(t, y(t; r))]\}$$

$$k_{3,1}(t, y(t; r)) = \min\{h f(t + \tfrac{h}{2}, u) \backslash u \in [z_{2,1}(t, y(t; r)), z_{2,2}(t, y(t; r))]\}$$
$$k_{3,2}(t, y(t; r)) = \max\{h f(t + \tfrac{h}{2}, u) \backslash u \in [z_{2,1}(t, y(t; r)), z_{2,2}(t, y(t; r))]\} \quad (4.3)$$
$$k_{4,1}(t, y(t; r)) = \min\{h f(t + ((1/4) + (-3/4) + (3/2)) h, u) \backslash u \in [z_{3,1}(t, y(t; r)), z_{3,2}(t, y(t; r))]\}$$
$$k_{4,2}(t, y(t; r)) = \max\{h f(t + ((1/4) + (-3/4) + (3/2)) h, u) \backslash u \in [z_{3,1}(t, y(t; r)), z_{3,2}(t, y(t; r))]\}$$

where in the fourth order Runge-Kutta method based on Contraharmonic Mean

$$z_{1,1}(t, y(t; r)) = y_1(t; r) + \tfrac{1}{2} k_{1,1}(t, y(t; r))$$
$$z_{1,2}(t, y(t; r)) = y_2(t; r) + \tfrac{1}{2} k_{1,2}(t, y(t; r)) \quad (4.4)$$
$$z_{2,1}(t, y(t; r)) = y_1(t; r) + \tfrac{1}{8} k_{1,1}(t, y(t; r) + \tfrac{3}{8} k_{2,1}(t, y(t; r)))$$
$$z_{2,2}(t, y(t; r)) = y_2(t; r) + \tfrac{1}{8} k_{1,2}(t, y(t; r) + \tfrac{3}{8} k_{2,2}(t, y(t; r)))$$
$$z_{3,1}(t, y(t; r)) = y_1(t; r) + \tfrac{1}{4} k_{1,1}(t, y(t; r) - \tfrac{3}{4} k_{2,1}(t, y(t; r) + \tfrac{3}{2} k_{3,1}(t, y(t; r))$$
$$z_{3,2}(t, y(t; r)) = y_2(t; r) + \tfrac{1}{4} k_{1,2}(t, y(t; r) - \tfrac{3}{4} k_{2,2}(t, y(t; r) + \tfrac{3}{2} k_{3,2}(t, y(t; r))$$

Define

$$F[t, y(t; r)] = \frac{k_{1,1}^2(t, y(t; r)) + k_{2,1}^2(t, y(t; r))}{k_{1,1}(t, y(t; r)) + k_{2,1}(t, y(t; r))} + \frac{k_{2,1}^2(t, y(t; r)) + k_{3,1}^2(t, y(t; r))}{k_{2,1}(t, y(t; r)) + k_{3,1}(t, y(t; r))} + \frac{k_{3,1}^2(t, y(t; r)) + k_{4,1}^2(t, y(t; r))}{k_{3,1}(t, y(t; r)) + k_{4,1}(t, y(t; r))} \quad (4.5)$$

$$G[t, y(t; r)] = \frac{k_{1,2}^2(t, y(t; r)) + k_{2,2}^2(t, y(t; r))}{k_{1,2}(t, y(t; r)) + k_{2,2}(t, y(t; r))} + \frac{k_{2,2}^2(t, y(t; r)) + k_{3,2}^2(t, y(t; r))}{k_{2,2}(t, y(t; r)) + k_{3,2}(t, y(t; r))} + \frac{k_{3,2}^2(t, y(t; r)) + k_{4,2}^2(t, y(t; r))}{k_{3,2}(t, y(t; r)) + k_{4,2}(t, y(t; r))}$$

The exact and approximate solutions at tn, 0 ≤ n ≤ N are denoted by $[Y(t_n)]_r = [Y_1(t_n; r), Y_2(t_n; r)]$ and $[y(t_n)]_r = [y_1(t_n; r), y_2(t_n; r)]$ respectively. The solution is calculated by grid points at (2.13).

By (4.1) and (4.5), we have

$$Y_1(t_{n+1}; r) \approx Y_1(t_n; r) + \tfrac{1}{3} F[t_n, Y(t_n; r)] \quad (4.6)$$
$$Y_2(t_{n+1}; r) \approx Y_2(t_n; r) + \tfrac{1}{3} G[t_n, Y(t_n; r)]$$

We define

$$y_1(t_{n+1}; r) \approx y_1(t_n; r) + \tfrac{1}{3} F[t_n, y(t_n; r)] \quad (4.7)$$
$$y_2(t_{n+1}; r) \approx y_2(t_n; r) + \tfrac{1}{3} G[t_n, y(t_n; r)]$$

The following lemmas will be applied to show convergence of these approximates in theorem 4.6. That is

Lemma 4.1 [17] Let the sequence of numbers satisfy

$$|W_{n+1}| \le A \ |W_n| + B, \ 0 \le n \le N - 1,$$

for some given positive constants A and B. Then

$$|W_n| \le A^n \ |W_0| + B \ \frac{A^n - 1}{A - 1}, \ 0 \le n \le N.$$

Lemma 4.2 [17] Let the sequence of numbers, satisfy

$$|W_{n+1}| \ \le \ |W_n| + A \ \max\{ \ |W_n|, \ |V_n| \} + B,$$
$$|V_{n+1}| \ \le \ |V_n| + A \ \max\{ \ |W_n|, \ |V_n| \} + B$$

for some given positive constants A and B, then denoting

$$U_n = |W_n| + |V_n|, \ 0 \le n \le N.$$

Then

$$U_n \le \bar{A}^n U_0 + \bar{B} \frac{\bar{A}^n - 1}{A - 1}, 0 \le \mathrm{n} \le \mathrm{N}$$

where $\bar{A} = 1 + 2A$ and B = 2B.

Let F(t, u, v) and G(t, u, v) be obtained by substituting $[y(t)]_r = [u, v]$ in (4.5)

$$F[t, \mathrm{u}, \mathrm{v}] = \frac{k_{1,1}^2(t, \mathrm{u}, \mathrm{v}) + k_{2,1}^2(t, \mathrm{u}, \mathrm{v})}{k_{1,1}(t, \mathrm{u}, \mathrm{v}) + k_{2,1}(t, \mathrm{u}, \mathrm{v})} + \frac{k_{2,1}^2(t, \mathrm{u}, \mathrm{v}) + k_{3,1}^2(t, \mathrm{u}, \mathrm{v})}{k_{2,1}(t, \mathrm{u}, \mathrm{v}) + k_{3,1}(t, \mathrm{u}, \mathrm{v})} + \frac{k_{3,1}^2(t, \mathrm{u}, \mathrm{v}) + k_{4,1}^2(t, \mathrm{u}, \mathrm{v})}{k_{3,1}(t, \mathrm{u}, \mathrm{v}) + k_{4,1}(t, \mathrm{u}, \mathrm{v})}$$

$$G[t, \mathrm{u}, \mathrm{v}] = \frac{k_{1,2}^2(t, \mathrm{u}, \mathrm{v}) + k_{2,2}^2(t, \mathrm{u}, \mathrm{v})}{k_{1,2}(t, \mathrm{u}, \mathrm{v}) + k_{2,2}(t, \mathrm{u}, \mathrm{v})} + \frac{k_{2,2}^2(t, \mathrm{u}, \mathrm{v}) + k_{3,2}^2(t, \mathrm{u}, \mathrm{v})}{k_{2,2}(t, \mathrm{u}, \mathrm{v}) + k_{3,2}(t, \mathrm{u}, \mathrm{v})} + \frac{k_{3,2}^2(t, \mathrm{u}, \mathrm{v}) + k_{4,2}^2(t, \mathrm{u}, \mathrm{v})}{k_{3,2}(t, \mathrm{u}, \mathrm{v}) + k_{4,2}(t, \mathrm{u}, \mathrm{v})}$$

The domain of F and G is

K = {(t, u, v) |0 ≤ t ≤ T, −∞ < v < ∞ , −∞ < u ≤ v}.

Theorem 4.1. Let F (t, u, v) and G (t, u, v) belong to C^4 (K) and let the partial derivatives of F and G be bounded over

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K. Then, for arbitrary fixed r, $0 \le r \le 1$, the approximate solutions (4.7) converge to the exact solutions Y_1 (t; r) and Y_2 (t; r) uniformly in t.

Proof: It is sufficient to show

where $t_N = T$. For n = 0, 1,…, N − 1, by using Taylor theorem we get

$$Y_1(t_{n+1};\ r) = Y_1(t_n;\ r) + \tfrac{1}{3}F[t_n,\ y(t_n;\ r)] + \frac{1333}{23040}h^5ML^4 + O(h^6) \quad (4.8)$$

$$Y_2(t_{n+1};\ r) = Y_2(t_n;\ r) + \tfrac{1}{3}G[t_n,\ y(t_n;\ r)] + \frac{1333}{23040}h^5ML^4 + O(h^6)$$

Let

$$W_n = Y_1(t_n;\ r) - y_1(t_n;\ r),$$
$$V_n = Y_2(t_n;\ r) - y_2(t_n;\ r)$$

Hence from (4.7) and (4.8)

$$W_{n+1} = W_n + \tfrac{1}{3}\{F[t_n,\ Y_1(t_n;\ r),\ Y_2(t_n;\ r)] - F[t_n,\ y_1(t_n;\ r),\ y_2(t_n;\ r)]\} + \frac{1333}{23040}h^5ML^4 + O(h^6)$$

$$V_{n+1} = V_n + \tfrac{1}{3}\{G[t_n,\ Y_1(t_n;\ r),\ Y_2(t_n;\ r)] - G[t_n,\ y_1(t_n;\ r),\ y_2(t_n;\ r)]\} + \frac{1333}{23040}h^5ML^4 + O(h^6)$$

Then

$$|W_{n+1}| \le |W_n| + \tfrac{2}{3}P\ h\ \max\{\ |W_n|,\ |V_n|\ \} + \frac{1333}{23040}h^5ML^4 + \mathrm{O}(\mathrm{h}^6)$$

$$|V_{n+1}| \le |V_n| + \tfrac{2}{3}P\ h\ \max\{\ |W_n|,\ |V_n|\ \} + \frac{1333}{23040}h^5ML^4 + \mathrm{O}(\mathrm{h}^6)$$

for $t \in [0, T]$ and P > 0 is a bound for the partial derivatives

of F and G. Thus by Lemma 4.2,

$$|W_n| \le \left(1 + \tfrac{4}{3}ph\right)^n |U_0| + \left(\frac{1333}{11520}h^5ML^4 + O(h^6)\right)\left(\frac{(1+\frac{4}{3}Ph)^n - 1}{\frac{4}{3}Ph}\right)$$

$$|V_n| \le \left(1 + \tfrac{4}{3}ph\right)^n |U_0| + \left(\frac{1333}{11520}h^5ML^4 + O(h^6)\right)\left(\frac{(1+\frac{4}{3}Ph)^n - 1}{\frac{4}{3}Ph}\right)$$

where $|U_0| = |W_0| + |V_0|$. In particular

$$|W_N| \le \left(1 + \tfrac{4}{3}ph\right)^N |U_0| + \left(\frac{1333}{15360}h^4ML^4 + O(h^5)\right)\left(\frac{(1+\frac{4}{3}Ph)^{\frac{T}{h}} - 1}{\frac{4}{3}P}\right)$$

$$|V_N| \le \left(1 + \tfrac{4}{3}ph\right)^N |U_0| + \left(\frac{1333}{15360}h^4ML^4 + O(h^5)\right)\left(\frac{(1+\frac{4}{3}Ph)^{\frac{T}{h}} - 1}{\frac{4}{3}P}\right)$$

Since $W_0 = V_0 = 0$, we obtain

$$|W_N| \le \frac{1333}{15360}h^4ML^4\left(\frac{e^{\frac{4}{3}pT} - 1}{p}\right) + O(h^5)$$

$$|V_N| \le \frac{1333}{15360}h^4ML^4\left(\frac{e^{\frac{4}{3}pT} - 1}{p}\right) + O(h^5)$$

And if $h \to 0$ we get $W_N \to 0$ and $V_N \to 0$ which completes the proof.

5. Numerical Examples

Example 5.1. Consider the fuzzy differential equation

$$\begin{cases} y'(t) = y(t),\ t \in [0,\ 1] \\ y(0) = (0.8{+}0.125\mathrm{r},\ 1.1{+}0.1\mathrm{r}) \end{cases} \quad (5.1)$$

The exact solution is given by

$Y(t;r) = [(0.8{+}0.125\mathrm{r})e^t,\ (1.1{+}0.1\mathrm{r})e^t],\ 0 \le r \le 1.$

At t =1 we get

$$Y(1;\ r) = [(0.8{+}0.125\mathrm{r})e^1,\ (1.1{+}0.1\mathrm{r})e^1],$$

Table 1 shows that the approximate, exact and error values calculated by the fourth order RKCoM method.

The graphical representation of the calculated and exact values of the fourth order Runge-Kutta method based on Contraharmonic mean using trapezoidal fuzzy number is given in figure.1.

Table. 1

r	t	FIVPRKCoM4		Exact		Error	
		y1	y2	Y1	Y2	y1	y2
0	1	2.174628	2.990114	2.174625	2.99011	2.99E-06	4.12E-06
0.2	1	2.242586	2.935748	2.242583	2.935744	3.09E-06	4.04E-06
0.4	1	2.310543	2.881383	2.31054	2.881379	3.18E-06	3.97E-06
0.6	1	2.3785	2.827017	2.378497	2.827013	3.27E-06	3.89E-06
0.8	1	2.446457	2.772651	2.446454	2.772647	3.37E-06	3.82E-06
1	1	2.514414	2.718286	2.514411	2.718282	3.46E-06	3.74E-06

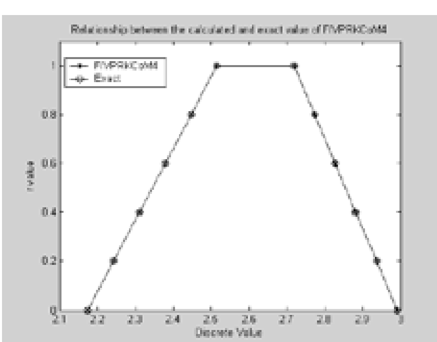

Figure 1

Example 5.2. Consider the fuzzy differential equation

$$\begin{cases} y'(t) = t\ y(t),\ t \in [0,\ 1] \\ y(0) = (\ (0.8{+}0.125\mathrm{r}),\ (1.1{+}0.1\mathrm{r})) \end{cases} \quad (5.2)$$

The exact solution is given by

$0 \le r \le 1.\ Y(t;\ r) = [\ (0.8{+}0.125\mathrm{r})e^{\frac{t^2-1}{2}},\ (1.1{+}0.1\mathrm{r})e^{\frac{t^2-1}{2}}\ ],$

The graphical representation of the calculated and exact values of the fourth order Runge-Kutta method based on Contraharmonic mean using trapezoidal fuzzy number is given in figure.2.

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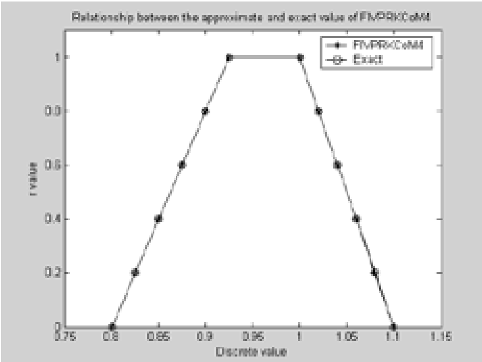

Figure 2

6. Conclusions

The proposed Fourth Order Runge-Kutta method based on Contraharmonic Mean has been applied in this paper for finding the numerical solution of fuzzy differential equations. In this procedure the fourth order Runge – Kutta method based on Contraharmonic mean is applied to solve a linear and nonlinear FDEs using the trapezoidal fuzzy number. Taking into account the convergence order for the proposed method, a higher order of convergence $O(h^4)$ is obtained. The Comparison of solutions of examples 5.1 and 5.2 from the tables and the figures show that the approximate solution of the fourth order Runge – Kutta method based on Contraharmonic Mean almost coincides with the exact solution when taking the step size h to be 0.1 itself. So, the fourth order Runge – Kutta method based on Contraharmonic Mean suits very well to solve a linear and nonlinear FDEs.

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