RESEARCH PAPER | Mathematics | Volume : 3 | Issue : 7 | July 2013 | ISSN - 2249-555X

Modeling and Simulation of Modified SIR Epidemic Model with Immigration and Non-monotonic Incidence Rate under Treatment

KEYWORDS: Differential Equation, Epidemiology, Immigration, Treatment function, Basic Reproductive Number, Endemic, Stability

D. Jasmine
PG & Research Department of Mathematics
Bishop Heber College, Trichy

E. C. Henry Amirtharaj
PG & Research Department of Mathematics
Bishop Heber College, Trichy

ABSTRACT *An epidemic model with non-monotonic incidence rate under a limited resource for treatment is proposed to understand the effect of the capacity of the treatment. This model is investigated by considering modified non-monotonic incidence rate with immigration. It is assumed that the treatment rate is proportional to the number of infective when it is below the capacity and is a constant when the number of infective is larger than the capacity. Existence and stability of the disease free and endemic equilibrium are investigated for both the cases. Some numerical simulations are given to illustrate the analytical results.*

1. Introduction

Mathematical modeling is an important tool to understand and predict the spread of infectious diseases. In this process, rate of incidence plays a crucial role. The incidence is an epidemiological model is the rate at which susceptible become infectious. Bilinear and std. incidence rate have been frequently used in classical epidemic models Capasso and Serio [2] introduced a saturated incidence rate g(I)S into epidemic models, where g(I) tends to a saturation level when I gets large, i.e., g(I) = KI / (1+αI). Mena Lorca and Hethcote [6] also analyzed an SIRS model with the same saturation incidence. Several different incidence rates have been proposed by other researchers. Nonlinear incidence rate of the form bI^pS^q were investigated by Liu. et. al. [9]. A very general form of non-linear incidence rate was considered by Derrick and Driessche [3]. Ruan and Wang [11] studied an epidemic model with a specific non-linear incidence rate $kI^2S / (1+\alpha I^2)$ and presented a detailed qualitative and bifurcation analysis of the model. A more general incidence $\lambda I^pS / (1+\alpha I^q)$ was proposed by many other researchers [1,4,8,10,12,s13]. Xiao and Ruan [14] proposed an epidemic model with non-monotonic incidence rate $\lambda IS / (1+\alpha I^2)$. Besides the rate and nature of incidence, treatment plays an important role to control the spread of diseases. This model is investigated and analyzed by Kar and Batabyal [7]. Also this model is modified by Gajendra Ujjainkar. et. al. [5]. Wang [12] proposed a treatment function

$T(I) = rI$, if $0 \le I \le I_0$

K_1, if $I > I_0$ (1)

where $K_1 = rI_0$ for some fixed value I_0.Kar and Batabyal [9] proposed a SIR model with non-monotonic incidence rate suggested by Xiao and Ruan [1] incorporating the above treatment function and non-monotonic incidence rate under a treatment function.

2. The Mathematical Model

Following Gajendra, Gupta, Sing and Khandelwal [10], the proposed model is

$$\frac{ds}{dt} = a - ds - \frac{\lambda IS}{1+\alpha_1 I+\alpha_2 I^2} + \beta R + \mu \quad (2)$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha_1 I+\alpha_2 I^2} - (d+m)I - T(I) \quad (3)$$

$$\frac{dR}{dt} = mI - (d+\beta)R + T(I) \quad (4)$$

where S(t), I(t), R(t) denote the number of susceptible, infective, recovered individuals, respectively. a is the recruitment rate of the population, d is the natural death rate of the population, λ is the proportionality constant, m is the natural recovery rate of the infective individuals, β is the rate at which recovered individuals lose immunity and return to susceptible class, μ the increase of at a constant rate and α_1 and α_2 are the parameter measures of the psychological or inhibitory effects and all other parameters have the same meaning as given in [9]. In this work take the treatment function T(I) defined by

$T(I) = rI$, if $0 \le I \le I_0$

K_1, if $I > I_0$

This means that the treatment rate is proportional to the infective when the number of infective is less or equal to some fixed value I_0 and the treatment is constant when the number of infective crosses the fixed value I_0. In the next section, the stability of the model taking two different cases of treatment function is discussed.

Case 1 : SIR model with $0 \le I \le I_0$

3. Equilibrium Points and Stability

When T(I) = r I, the model reduces to

$$\frac{ds}{dt} = a - ds - \frac{\lambda IS}{1+\alpha_1 I+\alpha_2 I^2} + \beta R + \mu \quad (5)$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha_1 I+\alpha_2 I^2} - (d+m+r)I \quad (6)$$

$$\frac{dR}{dt} = (m+r)I - (d+\beta)R \quad (7)$$

When the time derivatives equal to zero, we get a disease free equilibrium $E_0(a+\mu/d, 0,0)$. For the endemic equilibrium E*(S*,I*,R*) is the solution of

Lemma 3.1

S(t) + I(t) + R(t) = (a + μ)/d is an invariant manifold of the system attracting the first octant

Proof :

Let N(t) = S(t) + I(t) + R(t), then dN(t)=a-dN(t). Simple mathematical calculation shows that N(t) tends to (a+μ) / d as t tends to infinity.

Theorem 3.2

(i) When the basic reproductive number $R_0 \le 1$, there exist no positive equilibrium of the system (14)–(15), and in

Volume : 3 | Issue : 7 | July 2013 | ISSN - 2249-555X

that case the only disease free equilibrium (0,0) is a stable node.

(ii) When $R_0 > 1$, there exists a unique positive equilibrium of the system (14)-(15) and in that case (0, 0) is an unstable saddle point. Also the condition for which the unique positive equilibrium will be locally stable if $x^* < P_4 / P_3$.

Theorem: 3.4

When $K > u_1+c_1+cv_1$ and $u_1v_1 \leq cv_2$ the system has two positive equilibrium points (x_1, y_1) and (x_2, y_2), where x_1 and x_2 are the solution of the equation (27) under the parametric conditions given by (29) and when the conditions (32) and (35) are fulfilled at some equilibrium point that point must be asymptotically stable.

4. Numerical Simulation :

Case (i) :

When the treatment rate is ∞ to the infective so that $0 \leq I \leq I_0$. We choose the parameters in the model as follows.

a=3, d=0.1, α_2=0.1.β=0.1, λ = 0.3, μ= 0.1, m=0.01, r = 0.2 and $\alpha_1$varies from 0 to 4

Here the basic reproduction number R_0 = 30 >1. The equilibrium position goes lower and when the new parameter $\alpha_1$increases. Simultaneously when we increase the treatment function r we see that at α_1 =4 and r=1.5 susceptible population increases significantly.

When α_1=1, α_2=1, a= 15, β=0.5,d=2.5, m=10, r=0.01, μ=0.3, λ= 0.3 we have R_0=0.14676 < 1. In this case the disease dies out. Consider the values of the parameters

a=3, d=0.1, α_1=0.5, α_2 = 0.5, β=0.5, μ=0.1, λ=0.3, m=0.01, r=0.2

By rescaling the system we see that u–K <1 hence there exists the unique positive equilibrium x*= 8.3241 and y*=8.7403. For the above parameters p_4/p_3= 9.9240 and therefore then sufficient condition for stability $x^* < p_4/p_3$ is satisfied. Hence the point is locally stable.

Case (ii) :

When $I > I_0$, the parameters are

a=2.8, d=0.0453, α_2= 2, β= 0.13, λ=0.4, m = 0.01, K_1=0.7, μ=0.1with α_1=1

In this case S+I+R = $\frac{a+\mu}{d}$ = 64.01766 is invariant manifold. The system reduces to

$$\frac{dI}{dt} = \frac{0.4I(68.4327 - I - R)}{1 + I + 2I^2} - (0.0553)I - 0.7$$

$$\frac{dR}{dt} = (0.01)I - (0.1753)R + 0.7$$

The rescaling system reduces to

$$\frac{dx}{dT} = \frac{x(156.1498 - x - y)}{1 + 0.43825I + 0.384126I^2} - 0.315459x - 9.1116$$

$$\frac{dy}{dT} = 0.057045x - y + 9.1116$$

Here (u-K) < 0, and hence there exists unique positive equilibrium point (x*,y*) where x* = 8.3241 and y* = 8.7403. For the above choice of parameters p_3 = 392.8589 > 0, p_4 = 3651.5791, p_4/p_3 = 9.2949 and therefore the sufficient condition for local stability. i.e., $x^* < p_4 / p_3$ is satisfied.

In order to see the equilibrium point we have $u_1+c < (K–cv_1)$ and $u_1v_1 < cv_2$. Thus the above equation has two positive roots. It can be easily calculated that increasing values of α_1 slightly lowers down (x_1, y_1) where as pushes up the pair (x_2, y_2).

5. Concluding Remarks :

In this paper we see that the basic reproductive number plays an important role to control the disease. When $R_0 \leq 1$, there exists no positive equilibrium, and in that case the disease free equilibrium is globally stable, that is the disease dies out. But when R_0 >1, the unique endemic equilibrium is globally stable under some parametric condition. Also we see that the treatment rate plays a major role to control the disease. When μ= 0, the model coincides with that of Gajendra. et. al. [10].

REFERENCE

1. Anderson R.M, May R.N : Population Biology of Infectious Disease, I. Nature, (1979) 316-367. | 2. Capasso V and Serio G : A Generalization of the Kermack-Mckindrick Deterministic Epidemic Model, Math,Bio.Sci.42 (1978), 43-61. | 3. Derrick W.R and Van Der Driessche P : A Disease Transmission Model in a Non-constant Population, J. Math. Biol. 31, (1993), 495-512. | 4. Esteva L and Matias M.A : Model for Vector Transmitted Diseases with Saturation Incidence. Journal of Biological Systems, 9(4)(2001) 235-245. | 5. Gajendra Ujjainkar, Gupta V.K, Singh B, Khandelud R : An Epidemic Model with Modified Non-monotonic Incidence Rate under Treatment, Vol. 6 (2012) 1159-1171. | 6. Hethcote H.W and Van Den Driessch P : Some Epidemiological models with Non-linear Incidence . J. Math. Boil.29 , (1991) 271-287. | 7. Kar T.K and Ashim Batabyal : Modeling and Analysis of an Epidemic Model with Non-monotonic Incidence Rate under Treatment, Journal of Mathematical Research Vol.2, No. 1, (2010) 103-115. | 8. Liu W.M, Hethcote H.W and Levin S : A Dynamical Behavior of Epidemiological Models with Non-linear Incidence Rates, J. Math. Biol. 25, (1987) 359-380. | 9. Liu W.M, Levin S.A and Iwasa Y : Influence of Non-linear Incidence Rates upon the Behavior of SIRS Epidemiological Models. J. Math, Biol. (1986) 187-204. | 10. Mena Lorca J and Hethcote H.W : Dynamic Models of Infection Diseases as Regulators of Population Size. J.Math Bio. 30 (1992) 693-716. | 11. Ruan S and Wang W: Dynamical Behavior of an Epidemic Model with Non-linear Incidence Rate. J.Differential Equations, 188, (2003) 135-163. | 12. Wang W : Backward Bifurcation of an Epidemic Model with Treatment, J. Math, Bio. Sci. 201, (2006) 58-71. | 13. Wu.Z.Ferg L : Homoclinic Bifurcation in an SIQR Model for Childhood Diseases, J.Differ, Equat.(2000) 168-150. | 14. Xiao D and Ruan S : Global Analysis of an Epidemic Model with Non-monotone Incidence Rate. Math. Bio.Sci. 208, (2007) 419-429.