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Modern Optimization Model on Interval Estimation

KEYWORDS: INTERVAL ESTIMATED LINEAR PROGRAMMING MODEL (IELPM), LINEAR PROGRAMMING MODEL (LPM), INTERVAL VALUED FUNCTION (IVF), CONFIDENCE INTERVAL (CI).

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ABSTRACT *Optimization methods have been widely applied in statistics. This paper focuses the lower and upper values of interval estimated linear programming model (IELPM) are obtained by using the method of estimation. An IELPM is a linear programming model (LPM) with interval form of the coefficients in the objective function and all requirements. In conventional mathematical programming, the coefficients of the models are always treated as deterministic values. However uncertainty always exits in practical engineering models. Therefore, IELPMs may provide an alternative choice for considering the uncertainty into the optimization models. The solution of the IELPM is analyzed.*

1. Introduction

The optimization models have widely applied to many research fields. In mathematical programming, the coefficients of the models are always categorized as deterministic values. However uncertainty always exits in realistic models. There are several approaches to model uncertainty in optimization models. Though, interval estimated optimization models may provide opt for considering the uncertainty into IELPMs. That is, the coefficients in the objective function and all requirements are interval form and the coefficients in the IELPMs are assumed as closed intervals. LPMs with interval coefficients have been analyzed by many researchers, such as Chinneck and Ramadan (2000), Dantzig (1955), Herry Suprajitno and Ismail bin Mohd (2010), Kuchta (2008). Hladik (2007) has computed exact range of the optimal value for LPM in which input data can vary in some given real compact intervals, and he able to characterize the primal and dual solution sets, the bounds of the objective function resulted from two nonlinear programming models. Sengupta et al (2001) have reduced the interval number LPM into a bi-objective classical LPM and then obtained an optimal solution. Suprajitno and Mohd (2008) and Suprajitno et al (2009) presented some interval LPMs, where the coefficients and variables are in the form of intervals. Abbasi Molai and Khorram (2007) have introduced a Satisfaction Function (SF) to compare interval values on the basis of Tseng and Klein's idea and reduced the inequality constraints with interval coefficients in their satisfactory crisp equivalent forms and define a satisfactory solution to the model. The CIs have established to be useful tools for making inferences in many practical uncertain LPMs. The limits of uncertain data (i.e., determining the closed intervals to bind the possible observed data) are easier to be finding the CIs. The application of IELPMs are production planning, financial and corporate planning, health care and hospital planning, etc.,

The next section deals with review and some preliminaries on interval arithmetic. Succeedingly, the construction of CI was carried out. Then, the general model of IELPM and solving procedure are discussed. A numerical illustration with conclusion was discussed in the next section.

2. Preliminaries

In this section, necessary notations are discussed, which is useful for further consideration.

Let us denote by I the class of all closed and bounded intervals in P. If [a], [b] are closed and bounded intervals, we also adopt the notation $[a] = [\underline{a}, \bar{a}]$ and $[b] = [\underline{b}, \bar{b}]$, where $\underline{a}, \underline{b}$ and $\bar{a}, \bar{b}$ mean the lower and upper bounds of [a], [b]. Let $[a] = [\underline{a}, \bar{a}]$ and $[b] = [\underline{b}, \bar{b}]$ be in I. Then by definition,

(i) $[a] + [b] = [\underline{a}+\underline{b}, \bar{a}+\bar{b}] \subset I$

(ii) $[a] - [b] = [\underline{a}-\bar{b}, \bar{a}-\underline{b}] \in I$

(iii) $-[a] = [-\bar{a}, -\underline{a}] \in I$

(iv) $x[\underline{a}, \bar{a}] = [x\underline{a}, x\bar{a}]$, if $x \geq 0$

$[x\bar{a}, x\underline{a}]$, if $x \leq 0$

where x is a real number.

(v) An interval [a] is said to be positive, if $\underline{a} > 0$ and negative, if $\bar{a} < 0$.

(vi) If $[a] = [\underline{a}, \bar{a}]$ and also $[b] = [\underline{b}, \bar{b}]$ are bounded and real intervals, we define the multiplication of two intervals as follows:

$$[a][b] = [\min\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}]$$

1) If $0 \leq \underline{a} \leq \bar{a}$ and $0 \leq \underline{b} \leq \bar{b}$ then we have

$$[a][b] = [\underline{a}\underline{b}, \bar{a}\bar{b}]$$

2) If $0 \leq \underline{a} \leq \bar{a}$ and $\underline{b} < 0 < \bar{b}$ then we have

$$[a][b] = [\bar{a}\underline{b}, \bar{a}\bar{b}]$$

(vii) There are several approaches to define interval division. We define the quotient of two intervals as follows: Let $[a] = [\underline{a}, \bar{a}]$ and also $[b] = [\underline{b}, \bar{b}]$ be two nonempty bounded real intervals. Then if $0 \notin [\underline{b}, \bar{b}]$ we have

$$[a]/[b] = [\underline{a}, \bar{a}]\left[\frac{1}{\underline{b}}, \frac{1}{\bar{b}}\right] \qquad (2.3)$$

(viii) Power of interval for $n \in Z$ is given as:

When n is positive and odd or [a] is positive, then $[a]^n =$

$$\begin{cases} [\underline{a}^n, \bar{a}^n], & \text{if } \underline{a} \geq 0 \\ [\bar{a}^n, \underline{a}^n], & \text{if } \bar{a} < 0 \\ [0, \max\{(\underline{a})^n, (\bar{a})^n\}], & \text{otherwise} \end{cases}$$

When n is positive and even, then $[a]^n = [\underline{a}^n, \bar{a}^n]$,

When n is negative, $[a]^n = \frac{1}{[a]^n}$

(ix) For an interval [a] such that $\underline{a} \geq 0$, define the square root of [a] denoted by $\sqrt{[a]}$ as: $\sqrt{[a]} = \{\sqrt{b} : \underline{a} \leq b \leq \overline{a}\}$.

(x) Mid-point of an interval [a] is defined as $m([a]) = \frac{1}{2}(\underline{a} + \overline{a})$

(xi) Width of an interval [a] is defined as $w([a]) = \overline{a} - \underline{a}$.

(xii) Half-width of an interval [a] is defined as $w([a]) = \frac{1}{2}(\overline{a} - \underline{a})$.

Remark 2.1: Note that every real number a Î R can be considered as an interval [a, a] Î I.

Definition 2.1: The function F: R n ®I defined on the Euclidean space Rn called an Interval Valued Function (IVF) i.e., F(x) = F(x1, x2, …,,xn) is a closed interval in R. The IVF F can also be written as F(x) = $[\underline{F}(x)\ \overline{F}(x)]$, where $\underline{F}(x)$ and $\overline{F}(x)$ are real-valued functions defined on Rn and satisfy $\underline{F}(x) \leq \overline{F}(x)$ for every x Î R n. We say that the IVF F is differentiable at x0 Î R n if and only if the real–valued functions $\underline{F}(x)$ and $\overline{F}(x)$ are differentiable at x0.

Remark 2.2: Suppose A = $[\underline{a}, \overline{a}]$, B = $[\underline{b}, \overline{b}]$, then

1) F (A ³ B) > 0 Û $\overline{a} > \underline{b}$, 2) F (A > B) > 0 Û $\underline{a} > \underline{b}$ ٥ $\overline{a} > \overline{b}$,

3) F (A £ B) > 0 Û $\underline{a} < \overline{b}$, 4) F (A < B) > 0 Û $\underline{a} < \underline{b}$ ٥ $\overline{a} < \overline{b}$.

3. Construction of Confidence Interval

A general method of constructing CIs where distributed problems are avoided consists in using the Chebyshev inequality. If $\hat{\theta}$ is an estimator of q (not necessary unbiased), by Chebyshev inequality, for any positive e, we have

$$P_\theta\left[\hat{\theta} - \varepsilon\left\{E(\hat{\theta}-\theta)^2\right\}^{1/2} < \theta < \hat{\theta} + \varepsilon\left\{E(\hat{\theta}-\theta)^2\right\}^{1/2}\right] > 1 - \frac{1}{\varepsilon^2}$$

This gives us a CI for q with confidence level atleast 1-1/e2. This method can be used in all situations whether the random variable is discrete or continuous and the sample size is small or large.

3.1 Construction of confidence intervals in large samples

If L(q½x) is the likelihood function of a random sample of size n from a distribution with PDF f (x, q) and if the Fisher Information measure for the sample, viz., I (q), exists and if n is large then,

$$I_\theta(x) = E_\theta\left\{\left(\frac{\partial lnL}{\partial\theta}\right)^2\right\} = E_\theta\left\{\frac{-\partial^2 lnL}{\partial\theta^2}\right\}$$

If T is minimal sufficient for q , then $\frac{\partial lnL}{\partial\theta}\frac{1}{\sqrt{I_\theta(x)}}$ will be a function of T and q. So to construct CI for q, one can select, as a pivot

g(T, θ) = $\frac{\partial lnL}{\partial\theta}\frac{1}{\sqrt{I_\theta(x)}} \sim N(0,1)$ (3.1.1)

This pivot can be used to construct CI for the parameter q. For instance, for a (1 - α) CI we start with

$z_{\alpha/2} \leq g(T,\theta) \leq z_{\alpha/2}$ (3.1.2)

whose probability is (1 - α). If the two inequalities in (3.1.2) can be uniquely inverted to obtain inequaities for q, which is possible if g(T, q) is a monotonic function of q, then one can obtain (1 - α) CI for q.

4. General Model of Interval estimated Linear Programming Model (IELPM)

As was described in the previous section, an IELPM can be generated as follows:

Minimize f(x) = $\sum_{j=1}^{n}[\underline{c_j}, \overline{c_j}]x_j$

Subject to $\sum_{j=1}^{n}[\underline{a_{ij}}, \overline{a_{ij}}]\, x_j \leq\ \geq [\underline{b_i}, \overline{b_i}]$, (4.1)

xj ³ 0

Then we say that x = (x1, x2, …,xn) is a feasible solution of model (4.1) if and only if x1ai1 + …+ xjaij +…xnainÎ $[\underline{b_i}, \overline{b_i}]$ for all possible aij Î $[\underline{a_{ij}}, \overline{a_{ij}}]$, i = 1, 2,…, m and j = 1, 2,…, n. In other words, x = (x1, x2, …, xn) is a feasible solution of model (4.1) if and only if $\underline{b_i} \leq \sum_{j=1}^{n} a_{ij}x_j \leq \overline{b_i}$ for all possible aij Î $[\underline{a_{ij}}, \overline{a_{ij}}]$ i =1, 2,…, m and j = 1, 2,…, n. We adopt the notations $\underline{b_i} = (\underline{b_1}, \underline{b_2}, \ldots, \underline{b_m})$ and $\overline{b_i} = (\overline{b_1}, \overline{b_2}, \ldots, \overline{b_m})$. Also the feasible solution set S*= {xj Î Rn: $\sum_{j=1}^{n}[\underline{a_{ij}}, \overline{a_{ij}}]x_j \leq = \geq [\underline{b_i}, \overline{b_i}]$ and xj ³ 0} is assumed to be non empty and bounded.

5. An IELPM and its solving model

According to Abbasi Molai and Khorram (2007), the objective function of model (4.1) can be reduced into a linear three-objective LPM as follows:

Minimize {left limit of the interval objective function},

Minimize {right limit of the interval objective function},

Maximize {length of the interval objective function},

Subject {set of feasibility constraints}.

The principle of function f (x) indicates that for the minimization model, an interval with a smaller left and right limit value is inferior to an interval with a greater left and right limit value. Hence, in order to obtain the minimum of the interval objective function, considering the left and right limit value of the interval valued objective function is our primary concern. We reduce the interval objective function into a linear bi-objective function by its left and right limit value, i.e., the LPM with an interval objective function can be reduced into a LPM with a linear bi-objective function as follows:

Minimize {left limit of the interval objective function},

Minimize {right limit of the interval objective function} (5.1)

Subject to {set of feasibility constraints}.

We consider the length as a secondary attribute, only to confirm whether it is within the acceptable limit. If it is not, one has to increase the extent of length (uncertainty) according to his satisfaction and thus to obtain a longer interval among non-dominated alternatives. We can obtain the nondominated solutions via model (4.1). Model (4.1) can be expressed as simultaneously minimizing the left and right limit of the interval objective function. Here, a weighted function $\lambda_1\left(\sum_{j=1}^{n}\underline{c_j}x_j\right) + \lambda_2\left(\sum_{j=1}^{n}\overline{c_j}x_j\right)$ is introduced to obtain some non-dominated solutions, where l1 ≥ 0 and l2 ≥ 0 are the weights of the left and right end points of f (x), respectively, with l1 + l2 = 1. Taking l1 = 1 is regarded as an optimistic opinion of minimizing f (x) because the best situation is considered, whereas taking l2 = 1 is regarded as a pessimistic opinion because it is concerned with the worst situation. Considering that the decision maker is optimistic or pessimistic, we can reduce the linear bi-objective programming model (4.1) into a LPM, i.e., if the decision maker is optimistic, we will consider the following model:

Minimize {left limit of the interval objective function},

Subject o {set of feasibility constraints} (5.2)

Now, we will consider the inequality constraints of model (5.2).

Let A = $[\underline{a}, \overline{a}]$, B = $[\underline{b}, \overline{b}]$ and be a singleton variable. Now we can consider two possible states: i) Ax £ B, ii) Ax ³ B.

For Ax £ B, we propose an equivalent form of the interval inequality relation as follows:

Ax £ B Þ $\frac{F(\mathcal{A} \leq B) > 0}{\overline{F}(B < \mathcal{A}) \leq \alpha \in [0, 1]}$ (5.3)

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Where a may be interpreted as an assumed and fixed optimistic threshold by the decision maker. Now, according to the definition 2.1, remark 2.2, and relation (5.3) we obtain a equivalent form of the interval inequality relation as follows:

$$Ax \leq B \Rightarrow \begin{cases} \underline{a}\,x < \overline{b} \text{ (or equivalently } \overline{b} - \underline{a}\,x \geq \varepsilon) \\ \overline{a}\,x - \overline{b} \leq \alpha(\overline{b} - \underline{b}) + \alpha(\overline{a} - \underline{a})\,x, \end{cases}$$

Where a may be interpreted as an assumed and fixed optimistic threshold by the decision maker. e > 0 is a small positive value.

ii) Similarly, for Ax ≥ B, we propose an equivalent form of the interval inequality relation as follows:

$$Ax \geq B \Rightarrow \begin{cases} F(Ax \geq B) > 0 \\ \underline{F}(B > Ax) \leq \alpha \in [0, 1] \end{cases} \quad (5.4)$$

where a may be interpreted as an assumed and fixed optimistic threshold by the decision maker.

Similarly, we obtain a equivalent form of Ax≥B by the following pair:

$$Ax \geq B \Rightarrow \begin{cases} \overline{a}\,x < \underline{b} \text{ (or equivalently } \overline{a}\,x - \underline{b} \geq \varepsilon) \\ \underline{b} - \underline{a}\,x \leq \alpha(\overline{b} - \underline{b}) + \alpha(\overline{a} - \underline{a})\,x, \end{cases}$$

Where a may be interpreted as an assumed and fixed optimistic threshold by the decision maker. e > 0 is a small positive value.

From above procedure, mathematically we can write the model

Where a may be interpreted as an assumed and fixed optimistic threshold by the decision maker. e > 0 is a small positive value.

$$\text{Maximize } \underline{Z} = \sum_{j=1}^{n} \left(\overline{c}_j - \underline{c}_j\right) x_j,$$

$$\text{Subject to } \sum_{j=1}^{n} \underline{a}_j x_j \leq \overline{b}_i \left(\text{or equivalently } \sum_{j=1}^{n} \underline{a}_j x_j - \overline{b}_i \geq \varepsilon\right) \quad \forall i = 1, 2, \ldots, m \qquad (5.6)$$

$$\sum_{j=1}^{n} \overline{a}_j x_j - \overline{b}_i \leq \alpha(\overline{b}_i - \underline{b}_i) + \alpha \sum_{j=1}^{n} (\overline{a}_j - \underline{a}_j) x_j \quad \forall i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \forall j = 1, 2, \ldots, n,$$

6. Illustration

We consider a multiple period productions – smoothing model with interval shipping costs and preferring routes, crisp supplies and demands. Here, there is an example of using data obtained from confidence interval technique. Thus, the given IELPM can be written as the following

Maximize f(x) = [250.24, 501.180] x1 + [500.25, 751.10] x2

Subject to [100.48, 100.52] x1 + [100.21, 100.23] x2 ≤ [100.15, 100.75] (6.1)

[100.33, 100.38] x1 + [100.44, 100.46] x2 ≤ [100.30, 100.51]

xi ³ 0 " i= 1, 2.

Using the model (5.5), the optimization model can be formulated as

Maximize f(x) = 509.94 x1 + 259.85 x2

Subject to 100.48x1 + 100.21x2 ≤ 100.15

100.33x1 +100.44x2 ≤ 100.30 (6.2)

(100.52 - 0.04 α) x1 + (100.23 - 0.02 α) x2 ≤ 100.75 + 0.60 α

(100.38 - 0.05 α) x1 + (100.46 - 0.02 α) x2 ≤ 100.51 + 0.21 α

xi ³ 0 " i= 1, 2.

In model (6.2), a Î [0, 1] is assumed and fixed optimistic threshold by the decision maker. By using Excel Solver, the obtained the following optimal solution:

x1 = 0.209822, x2 = 0.789014, optimal solution is 250.5769

7. Conclusion

This paper focused about IELPM involving CI technique. As a general rule, solution methods based on statistical approximations are applicable for both small and large samples. Work is in progress to apply and check the approach for solving multiple period productions –smoothing model under interval data environment.

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