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An Implicit Difference Method for Solving Fractional Reaction-Dispersion Equation in the Grunwald Form

KEYWORDS: Fractional derivative, implicit Euler method, fractional reaction-dispersion equation, stability, convergence

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ABSTRACT *In this paper, a numerical solution of fractional reaction-dispersion equation has been presented. The algorithm for the numerical solution for this equation is based on implicit finite difference method. The consistency, unconditional stability, and convergence of the fractional order numerical method are described.*

The numerical method has been applied to solve a practical numerical example with comparing the results. The results were presented in tables using the MathCAD 12 software package when it is needed. The implicit finite difference method appeared to be effective and reliable in solving fractional reaction-dispersion equation.

1. Introduction

In recent years there has been a great deal of interest in fractional partial differential equations [1, 2, 3, 4, 5]. These equations arise quite naturally in continuous time random walk with spatial and temporal memories.

The most significant advantage of the fractional order models in comparison with integer-order models is based on its important fundamental physical considerations. However, because of the absence of appropriate mathematical methods, fractional-order dynamical systems were studied only marginally in theory and practice of control systems. Numerical methods and theoretical analyses of fractional differential equations are very difficult tasks [6, 7, 8].

The method discussed in this paper is an implicit finite difference method designed for solving fractional reaction-dispersion equation where the fractional derivative is in the shifted Grunwald estimate form. The unconditional stability and convergence of the implicit finite difference approximation are analyzed and finally, we will present some examples to show the efficiency of our numerical method.

2. Finite Difference Method for Solving the Fractional Reaction-Dispersion Equation

In this section, we use the implicit finite difference method for solving the fractional reaction-dispersion equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + D_x^{\alpha} u(x,t) \qquad (1)$$

In this problem initial and boundary conditions are considered which are:

$u(x,0)=f(x)$, $L < x < R$ (2)

$u(L,t) = \psi_1(t)$, $0 \le t \le T$ (3)

$u(R,t)=\psi_2(t)$, $0 \le t \le T$ (4)

where bounded space domain is [L,R], f is a known function of x, ψ_1 and ψ_2 are known functions of t. the fractional derivative of order α . And D_x^{α} are defined as the shifted Grunwald estimate to the α- the fractional derivative, [9]:

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = \frac{1}{(\Delta x)^{\alpha}} \sum_{k=0}^{i+1} g_k u_{i-k+1,j} + O(\Delta x) \qquad (5)$$

where $g_k = (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$, k=0,1,2,…

The finite difference method starts by dividing the x-interval [L, R] into n subintervals to get the grid points $x_i = L + i\Delta x$, where $\Delta x = (R-L)/n$ and i=0,1,…,n. Also, the t-interval [0,T] is divided into m subintervals to get the grid points $t_j = j\Delta t$, j = 0,1,…,m, where $\Delta t = T/m$.

Next, by evaluating eq.(1) at (x_i,t_j) and use the implicit Euler method one can get:

$$\frac{u(x_i,t_{j+1}) - u(x_i,t_j)}{\Delta t} = -u(x_i,t_j) + \frac{\partial^{\alpha} u(x_i,t_j)}{\partial x^{\alpha}} + O(\Delta t) \qquad (6)$$

Use fractional derivative of the shifted Grunwald estimate to α-the fractional derivative eq.(5), to reduce eq.(6) as in the following form:

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = -u_{i,j+1} + \frac{1}{(\Delta x)^{\alpha}} \sum_{k=0}^{i+1} g_k u_{i-k+1,j+1},$$

$$i = 1,2,\dots,n-1,\ j = 0,1,\dots,m-1 \qquad (7)$$

where $u_{i,j} = u(x_i,t_j)$.

The resulting equation can be implicitly solved for $u_{i,j+1}$ to give

$$u_{i,j+1} - \beta \sum_{k=0}^{i+1} g_k u_{i-k+1,j+1} = \eta u_{i,j},$$

$$i = 1,2,\dots,n-1,\ j = 0,1,\dots,m-1 \qquad (8)$$

where $\beta = \frac{\Delta t}{(1+\Delta t)(\Delta x)^{\alpha}}$, $\eta = \frac{1}{(1+\Delta t)}$.

Also form the initial condition and boundary conditions one can get

$u_{i,0} = f(x_i)$, i=0,1,…, n

$u_{L,j} = \psi_1(t_j)$, j=0,1,…, m

$u_{R,j} = \psi_2(t_j)$, j=0,1,…, m

By evaluating eq.(8) at each $i = 1,2,\dots,n-1$ and $j = 0,1,\dots,m-1$ and using the above three one equations one can get the numerical solutions of eq.(1) .

Theorem. The implicit finite difference method eq.(8) is unconditionally stable for all 1< α < 2.

Proof:
The system of equations defined by (8), together with the

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initial and boundary condition can be written in the implicit matrix form $\underline{C}\underline{U}_{j+1} = \eta \underline{U}_j$ where

$\underline{U}_j = [u_{0,j}, u_{1,j}, \ldots, u_{n,j}]^T$, and

$\underline{C}$ is the matrix of coefficients, and is the sum of a lower triangular matrix and a super diagonal matrix. Therefore the resulting matrix entries $C_{i,j}$ for $i = 1,2,\ldots,n-1$ and $j = 1,2,\ldots,n-1$ are defined by

$$C_{i,j} = \begin{cases} 1+\beta_i g_1 & for \quad j=i \\ -\beta_i g_2 & for \quad j=i-1 \\ -\beta_i g_0 & for \quad j=i+1 \\ -\beta_i g_{i-j+1} & for \quad j<i-1 \end{cases}$$

To illustrate this matrix pattern, we list the corresponding equations for the rows i =1, 2 and n-1:

$$-\beta_1 g_2 u_{0,j+1} + (1+\beta_1 g_1) u_{1,j+1} - \beta_1 g_0 u_{2,j+1} = \eta u_{1,j}$$

$$-\beta_2 g_3 u_{0,j+1} - \beta_2 g_2 u_{1,j+1} + (1+\beta_2 g_1) u_{2,j+1} - \beta_2 g_0 u_{3,j+1} = \eta u_{2,j}$$

$$-\beta_{n-1} g_n u_{0,j+1} \cdots - \beta_{n-1} g_2 u_{n-2,j+1} + (1+\beta_{n-1} g_1 u_{n-1,j+1}) - \beta_{n-1} g_0 u_{n,j+1} = \eta u_{n-1,j}$$

According to the Greshgorin theorem [10], the eigenvalue of the matrix $\underline{C}$ are in the disks centered at $C_{i,i} = 1 - \beta_i g_1 = 1 + \beta_i \alpha$ with radius

$$r_i = \sum_{\substack{k=0 \\ k\neq i}}^{n} C_{i,k} = \sum_{\substack{k=0 \\ k\neq i}}^{i+1} \beta_i g_k \le \beta_i \alpha$$

With strict inequality holding true when α is not an integer. This implies that the eigenvalue of the matrix $\underline{C}$ are all no less then 1 in magnitudes. Hence the spectral radius of the matrix $\underline{C}^{-1}$ is less than 1. Thus any error in $\underline{U}^j$ is not magnified, and therefore the implicit Euler method defined above is unconditionally stable.

3. Consistency, Stability and Convergent

The implicit Euler method defined by (8) is consistent with order $O(\Delta t) + O(\Delta x^{[\alpha]})$, where $[\alpha]$ denotes the largest integer that is less than or equal to α. That consistency of the finite difference method together with the above result on unconditionally stability implies that the implicit Euler method is convergent.

4. Numerical Example

In this section, we give some numerical results that confirm our theoretical.

Example: Consider the fractional reaction-dispersion equation:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} - x^{0.5} e^{-t}$$

subject to the initial condition

u (x,0) = $x^{0.5}$, 0 < x < 0.2

and the boundary conditions

u (0,t) = 0, 0 £ t £ 0.025

u (0.2,t) =0.44721 e^{-t}, 0 £ t £ 0.025

This fractional partial differential equation together with the above initial and boundary condition is constructed such that the exact solution is u(x, t)= $x^{0.5}$ e^{-t}.

Table1 show the numerical solution using the implicit finite difference approximation for $\Delta x =$ 0.05 and $\Delta t = 0.0125$.

Now if we assume initial condition and boundary condition for this problem

u (x,0) = $x^{0.5}$, 0 < x < 0.4

u (0,t) = 0, 0 £ t £ 0.02

u (0.4,t) = 0.63246 e-t, 0 £ t £ 0.025

Table 2 give the numerical solution using the implicit finite difference approximation for $\Delta x = 0.1$ and $\Delta t = 0.0125$.

From table 1 and 2, it can be seen that that good agreement between the numerical solution and exact solution.

Table 1: The numerical solution of example by using the finite difference method for $\Delta x =$ 0.05 and $\Delta t = 0.0125$

x	t	Numerical solution	Exact Solution	Error
0.05	0.0125	0.19300	0.22083	2.78300 E-2
0.10	0.0125	0.26500	0.31230	4.73000 E-2
0.15	0.0125	0.37400	0.38249	8.48720 E-3
0.05	0.0250	0.18900	0.21809	2.90860 E-2
0.10	0.0250	0.28100	0.30842	2.74200 E-2
0.15	0.0250	0.36900	0.37774	8.73600 E-3

Table 2: The numerical solution of example by using the finite difference method for $\Delta x = 0.1$ and $\Delta t = 0.0125$

x	t	Numerical solution	Exact Solution	Error
0.1	0.01	0.29700	0.31230	1.53000 E-2
0.2	0.01	0.40700	0.44166	3.46600 E-2
0.3	0.01	0.53700	0.53700	3.92000 E-3
0.1	0.02	0.28600	0.28600	2.24200 E-2
0.2	0.02	0.41000	0.41000	2.61700 E-2
0.3	0.02	0.52900	0.52900	5.20000 E-3

5. Conclusions

In this paper:

1- Numerical method for solving the fractional reaction-dispersion equation has been described and demonstrated.

2- The implicit Euler method is proved to be unconditionally stable and converges.

REFERENCE [1] Podlubny I.," Fractional Differential Equations", Academic Press, 1999. | [2] Huang F. and Liu F., "The Time Fractional Diffusion Equation and the Advection-Dispersion Equation", Anziam Journal, Vol. 46 (2005): 1-14. | [3] Zhuang P., Liu F., Anh V. and Turner I., "New Solution and Analytical Techniques of the Implicit Numerical Method for the Anomalous Sub-diffusion Equation", SIAM J. Numerical Analysis, Vol. 46(2)(2008): 1079-1095 | [4] Yuste S.B. and Acedo L., "An Explicit Finite Difference Method and a New Von Neumann-Type Stability Analysis for Fractional Diffusion Equations", SIAM J.Numer. Anal., Vol. 42(5)(2005): 1862-1874. | [5] Meerschaert M. and Tadjeran C., "Finite Difference Approximations for Two-Sided Space-Fractional Partial Differential Equations", Appl. Numer. Math., Vol. 56(2006): 80-90. | [6] Zhuang P., Liu F., Anh V. and Turner I., "Numerical Solution of Space Fractional Fokker-Planck Equation", J. Comp. and Appl. Math., Vol.166(2004): 209-219. | [7] Liu F., Anh V., Turner I. and Zhuang P., "Numerical Simulation for Solute Transport in Fractal Porous Media", ANZIAM J., Vol.45 (E)(2004): 461-473. | [8] Chen J., Liu F., Turner I. and Anh V.,"The Fundamental and Numerical Solutions of the Riesz Space Fractional Reaction-Dispersion Equation", ANZIAM J., 50(2008):45-57. | [9] Meerschaert M. and Tadjeran C., "Finite Difference Approximation for Fractional Advection Dispersion Flow Equations", J. Comp. Appl. Math., Vol. 172(2004): 65-77. | [10] Isaacson E. and Keller H.B., "Analysis of Numerical Methods", Wiley, New York, PP. 135-136, 1966.