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Acyclic Coloring of Helm Graph Families

KEYWORDS: Acyclic coloring, Middle graph, Central graph and Total graph.

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ABSTRACT *An acyclic coloring of a graph G is a proper vertex coloring (no two adjacent vertices of G have the same color) such that the induced subgraph of any two color classes is acyclic. The minimum number of colors needed to acyclically color the vertices of a graph G is called as acyclic chromatic number and is denoted by a(G). In this paper, we give the exact value of the acyclic chromatic number of Middle , Central and Total graph of Helm Graph families.*

1. INTRODUCTION

All graphs considered here are finite, simple and undirected. In the whole paper, the term coloring will be used to refer vertex coloring of graphs. A proper coloring of a graph G is a coloring of the vertices of G such that no two neighbors in G are assigned the same color.

1.1 Definition

A subgraph H of a graph G is said to be induced subgraph if it has all the edges that appear in G over the same vertex set. The subgraph induced by the vertex set {v1,v2,v3,…vk} is denoted by <v1,v2,v3,…,vk>.

1.2 Definition

A vertex coloring of a graph is said to be acyclic [9] if the induced subgraph of any two color classes is acyclic. In other words, the subgraph induced by any two color classes is a forest.

1.3 Definition

The minimum number of colors needed to acyclically color the vertices of a graph G is called its acyclic chromatic number and is denoted by a(G).

1.4 Definition

A cycle in a graph G is said to be a bicolored (j,k)-cycle if all its vertices are properly colored with two colors j and k. A graph G is said to be a (j,k)-cycle free graph if it do not have any bicolored (j,k)-cycle.

1.5 Definition

The Helm Hn, is the graph obtained from a Wheel graph Wn, by attaching a pendent edge at each vertex of the n-cycle.

In this paper, we obtain the exact value of the acyclic chromatic number of the Helm graph families.

2. ACYCLIC COLORING OF M(Hn)

Let G be a graph with vertex set V(G) and edge set E(G).

2.1 Definition

The Middle graph [2] ,denoted by M(G), of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining those pairs of these new vertices with edges which lie on adjacent edges of G.

In Helm Hn, let v be the root vertex and v1,v2,v3,….,vn be the vertices of n-cycle. Let w1,w2, w3, … wn be the n pendent vertices of Hn. Let ek(k=1 to n) be the newly added vertex on the edge joining v and vk and fk (k=1 to n) be the newly added vertex on the edge joining vk and vk+1. Let gk (k=1 to n) be the newly added vertex on the edge joining vk and wk. We use these notations for sections 3 and 4 also.

2.2 Structural properties of M(Hn).

By definition 2.1,M(Hn) has the following structural properties.

(i) < v,ek; k=1 to n> form a clique of order n+1.
(ii) For each k=2 to n, the neighbors of vk are {ek,fk,fk-1,gk} and the neighbors of v1 are {e1,f1,fn,g1}.
(iii) The neighbors of wk is {gk},k=1to n.
(iv) For each k=2 to n-1, the neighbors of fk are {fk-1,fk+1,ek,ek+1,vk,vk+1,gk,gk+1} and the neighbors of f1 and fn are respectively {fn,f2,e1,e2,v1,v2,g1,g2} and{fn-1,f1,en,e1,vn,v1,gn,g1}.
(v) For each k=1 to n, ek and gk are adjacent.

We use these structural properties, to find the acyclic chromatic number of M(Hn). Now, we present a coloring algorithm for M(Hn) and we prove that the coloring is acyclic in the immediate following theorem.

2.3 Coloring Algorithm of M(Hn), $n \geq 4$.

Input : M(Hn)

```
V←  {v,e1,e2,…,en,v1,v2,…,vn,f1,f2,…,fn,g1,g2,…,gn,w1,w2,…,wn}

E← {e1',e2',…en', eij'(1≤ i < j < n), e1",e2",…en", f1',f2',…fn', f1",f2",…fn",g1',g2',..,gn',

 g1",g2",…,gn", h1',h2',…,hn',h1",h2",…,hn", d1',d2',…dn', d1",d2",…dn" ,

 l1',l2',…,ln',l1",l2",…,ln"}

for k= 1 to n
 {
 vek← ek' ;
 }
end for
for j= 1 to n-1
 {
for k= 1 to n
{
if j < k,
ejek ← ejk' ;
}
}
end for
end for
for k= 1 to n
{
ekvk← ek" ; ekfk← fk' ; vkgk← gk' ; gkwk← gk" ; ekgk← dk" ;
}
end for
for k= 1 to n-1
{
fkek+1← fk" ;
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}
end for
fne1← fn" ;
for k= 1 to n-1
{
fk fk+1← hk' ; fk gk+1← hk" ; fk vk+1← lk" ;
}
end for
fnf1 ← hn' ; fng1 ← hn"; fn v1← ln" ;
for k= 1 to n
{
gkfk← dk' ; vkfk← lk' ;
}
end for
v←n+1;
for k= 1 to n
{
ek← k ;
}
end for
for k= 1 to n
{
vk← n+1; wk← n+1;
}
end for
for k= 1 to n
{
r← k+2;
if r ≤ n,
fk← r ;
else
fk← r-n ;
}
end for
for k= 1 to n
{
s← k+3;

if s ≤ n,
gk← s ;
else
gk← s-n ;
}
end for
```

2.4 Theorem
The acyclic chromatic number of M(Hn) is

a[M(Hn)]=n+1, n≥4.

Proof:
First, we prove that the coloring of M(Hn) is acyclic. For this, let us assign colors to the vertices of M(Hn), using algorithm2.3.

Case(i)
Consider the colors n+1 and k , k=1 to n. The color class of n+1 is {v,vj ,wj ;j=1 to n} whereas the color class of k is {ek ,fk-2,gk-3}. The induced subgraph of these color classes is a forest as it contain the bicolored disjoint paths v ek vk ,vk-2 fk-2 vk-1 and vk+2gk+2wk+2. Therefore, M(Hn) is (k,(n+1))-cycle free.

Case(ii)
Consider the color k and k+1,1≤ k ≤n-1. The color class of k is {ek ,fk-2,gk-3}whereas the color class of k+1 is {ek+1,fk-1,gk-2}. The induced subgraph of these color classes is a forest as it contain the bicolored path gk-2 fk-2 fk-1 ek ek+1 and an isolated vertex gk-3. Therefore, M(Hn) is (k,(k+1))- cycle free graph.

Case(iii)
Consider the colors j and k, 1≤ j,k ≤ n. The induced subgraph of the color classes of these colors is a forest as it contain the bicolored paths fk-2 ej ek and gk-3fj-2(when| j-k|=2) or the bicolored path of length 3 and isolated vertices (when| j-k | ≥2). Thus, M(Hn) is (j,k)-cycle free graph.

In all the three cases, the induced subgraph of any two color classes is acyclic and hence the coloring is acyclic.

As M(Hn) has a clique of order n+1, we need minimum n+1 colors for proper coloring

(see Fig.1). Therefore, a[M(Hn)]=n+1, n ≥ 4.

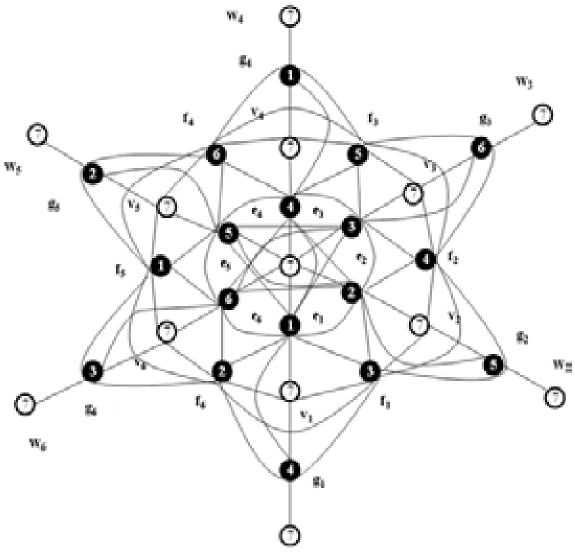

Fig.1.a[M(H6)] = 7

2.5 Remark
(i) a[M(H2)]=5

(ii)a[M(H3)]= 6.

3. ACYCLIC COLORING OF C(Hn)

3.1 Definition
Let G be a graph with vertex set V(G) and edge set E(G). The central graph of G, denoted by C(G)[11], is obtained from G by subdividing each edge exactly once and joining all the non adjacent vertices of G.

3.2 Structural properties of C(Hn)
(i) < v,wk; k=1 to n> form a clique of order n+1.
(ii) {v ,fk ,gk ;k=1 to n} form an independent set.
(iii) The neighbors of vk,(k= 2 to n-1) is {ek ,fk ,gk}∪{vj;j=1 to n and j≠k-1,k+1}∪{wj;j=1 to n and j≠k}. The neighbors of v1 is {e1,f1,g1) ∪{vj ;j= 3 to n-1}∪{wj; j= 2 to n} and that of vn is {en ,fn ,gn}∪{vj ;j= 2 to n-2}∪{wj; j= 1 to n-1}
(iv) The neighbors of ek is {v,vk},k=1 to n.
(v) The neighbors of gk is {vk,wk},k=1 to n.

We use these structural properties in the coloring algorithm of C(Hn) and we prove that the coloring is acyclic in the immediate following theorem.

3.3 Coloring Algorithm of C(Hn), n ≥ 5.
Input : C(Hn)
V← {v,e1,e2,…,en,v1,v2,…,vn,f1,f2,…,fn,g1,g2,…,gn,w1,w2,…,wn};
E← {e1',e2',…en', e1",e2",…en",l1',l2',…,ln' f1',f2',…fn', f_1",f_2",…f_n",g_1',g_2',..,g_n',g_1",g_2",…,g_n", d_{ij} (1 ≤ i,j ≤ n, i≠j), l_{ij}(1≤ i < j ≤ n), h_{ij}(1≤ i ≤ n-2, i < j ≤ n, j≠ i+1)};

```
for k= 1 to n
{
v ek← ek' ; ek vk← ek"; v wk← lk';
}
end for
for k= 1 to n
{
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vk fk← fk' ; vk gk← gk'; gk wk← gk";
}
end for
for k= 1 to n-1
{
if k < n,
fk vk+1 ← fk" ;
}
end for
fn v1 ← fn" ;
for j= 1 to n
{
for k= 1 to n
{
if j≠k,
vj wk ← djk ;
}
}
end for
end for
for j= 1 to n
{
for k= 1 to n
{
if j < k,
wj wk ← ljk ;
}
}
end for
end for
for k= 3 to n-1
{
v1 vk ← h1k ;
}
end for
for j= 2 to n-2
{
for k= j+2 to n
{
vj vk ← hjk ;
}
}
end for
end for
v← n+1;
for k= 1 to n
{
fk← n+1; gk← n+1;
}
end for
for k= 1 to n
{
wk← k ;
}
end for
for k= 1 to 2
{
vk← k ;
}
end for
for k= 3 to n
{
vk← n+k-1 ;
}
end for
for k= 1 to n m
{
r← k+1 ;
if r ≤ n,
ek← r ;
else
ek← r-n ;
}
end for
```

3.4 Theorem
The acyclic chromatic number of C(Hn) is a[C(Hn)]= 2n-1, n ≥ 5.

Proof
We prove the theorem by showing the coloring given in sec 3.3 is acyclic.

As the two neighbors of each fk (k= 1 to n) have different colors, any bicolored cycle cannot contain fk. The same argument is true for gk,3 ≤ k ≤ n. Similarly, any bicolored cycle cannot contain the path v ek vk (k= 1 to n), since the two neighbors of ek (k=1 to n) have different colors. Since the color class of k (n+2 ≤ k ≤ 2n-1) is a single vertex vk-n+1, any bicolored cycle cannot contain the vertices vj (3 ≤ j ≤ n). So, we discuss the following cases.

Case(i)
Consider the colors 1 and 2. The color class of 1 is {v1,w1,en} whereas the color class of 2 is {v2,w2,e1}. The induced subgraph contains only the bicolored path v1w2w1v2, as v1 and v2 are non adjacent. Thus, C(Hn) is (1,2)-cycle free.

Case(ii)
Consider the colors n+1 and k, k= 1,2. The induced subgraph contains the bicolored path en v w1 g1 v1 f1, when k=1 and the bicolored path e1 v w2 g2 v2 f2, when k=2. In both cases, C(Hn) is (k,n+1)-cycle free graph.

Case(iii)
Consider the colors 1 and k, 3 ≤ k ≤ n. The color class of 1 is {v1,w1,en} and that of k is {wk,ek-1}. The induced subgraph contains only the bicolored path v1wkw1 and therefore C(Hn) is (1,k)-cycle free.

Case(iv)
Consider the colors 2 and k, 3 ≤ k ≤ n. By the same argument as in case (iii) , C(Hn) is (2,k)-cycle free.

Case(v)
Consider the colors (n+1) and k,3≤ k ≤ n. The induced subgraph contains only the bicolored path ek-1v wk gk and hence C(Hn) is acyclic.

Case(vi)
Consider the colors j and k,3 ≤ j,k ≤ n. In this case, the induced subgraph contains only the bicolored edge wj wk and isolated vertices. So, C(Hn) is (j,k)-cycle free graph.
Thus, C(Hn) is acyclic.

As C(Hn) has a clique of order n+1, a[C(Hn)] ≥ n+1. The colors n+2 to 2n-1 are assigned respectively, to the vertices v3,v4,…,vn. If we assign the same color, say k, to the non adjacent vertices vi,vi+1(3≤ i ≤ n-1), then,w1viv1vi+1w1 form a bicolored (1,k)-cycle. So, different colors are assigned to the vertices v3,v4,…,vn. Thus, we need minimum 2n-1 colors for acyclically color the vertices of C(Hn) (see Fig.2) and hence, a[C(Hn)] = 2n-1, n ≥ 5.

Fig 2. a[C(H5)] = 9

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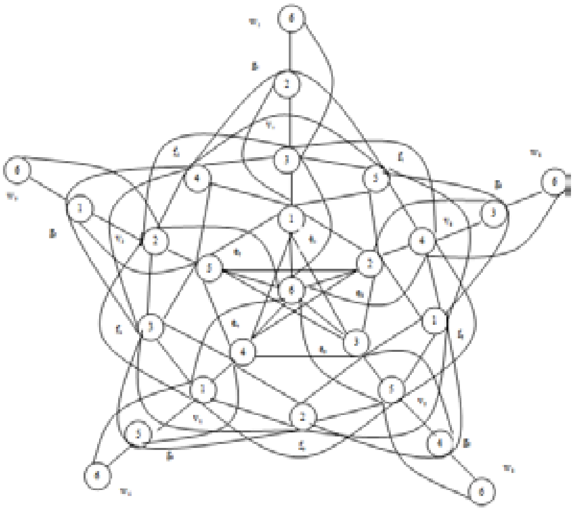

Fig 3. a[T(H5)] = 6

3.5 Remark

(i) a[C(Hn)]= 2n-1,n= 2,3.
(ii) a[C(H4)]= 6.

4. ACYCLIC COLORING OF T(Hn)

4.1 Definition

The Total graph [2] of a graph, denoted by T(G), is a graph such that the vertex set of T is V(G)UE(G) and two vertices are adjacent in T iff their corresponding elements are either adjacent or incident in G.

4.2 Structural properties of T(Hn)

By the definition of Total graph, T(Hn) has the following properties.

(i) <v,ek; k=1 to n> form a clique of order n+1.
(ii) The neighbors of vk (k=2 to n-1) is {v,ek ,vk-1,vk+1,fk-1,fk,gk,wk}. The neighbors of v1 and vn are respectively {v,e1,v2,vn,f1,fn,g1,w1} and {v,en,vn-1,v1,fn-1,fn,gn,wn}.
(iii) The neighbors of fk (k=2 to n-1) is {ek ,vk ,ek+1,vk+1,fk-1,fk+1,gk,gk+1}.The neighbors of f1 and fn are respectively {e1,v1,g1,e2,v2,g2,fn ,f2} and {en,vn,gn,e1,v1,g1,fn-1,f1}.
(iv) The neighbors of gk (k=2 to n-1) is {ek,vk,wk,fk-1,fk}. The neighbors of g1 and gn are respectively {e1,v1,w1,fn,f1} and {en,vn,wn,fn-1,fn}.
(v) The neighbors of wk is {gk ,vk}, k= 1 to n.

Now, we present the coloring algorithm of T(Hn) and then we prove that the coloring is acyclic in the immediate following theorem.

4.3 Coloring Algorithm of T(Hn),n≥ 5

Input: T(Hn)

```
V← {v,e1,e2,…,en,v1,v2,…,vn,f1,f2,…,fn,g1,g2,…,gn,w1,w2,…,wn}
E← {e1′,e2′,…en′, eij′(1≤ i < j < n), e1″,e2″,…en″, f1′,f2′,…fn′, f1″,f2″,…fn″,g1′,g2′,..,gn′,
g1″,g2″,…,gn″, h1′,h2′,…,hn′,h1″,h2″,…,hn″, d1′,d2′,…dn′, d1″,d2″,…dn″ ,
l1′,l2′,…,ln′,l1″,l2″,…,ln″,x1′,x2′,…,xn′,x1″,x2″,…,xn″, y1′,y2′,…,yn′}
for k= 1 to n
{
vek← ek′ ; vvk← xk′;
}
end for
for j= 1 to n-1
{
for k= 1 to n
{
if j < k,
ejek ← ejk ;
}
}
end for
end for
for k= 1 to n
{
ekvk← ek″ ; ekfk← fk′ ; vkgk← gk′ ;
vkwk ← xk″; gkwk← gk″ ; ekgk← dk″ ;
}
end for
for k= 1 to n-1
{
fkek+1← fk″ ;
}
end for
fne1← fn″ ;
for k= 1 to n-1
{
 vk vk+1← yk′ ; fk fk+1← hk′ ; fk gk+1← hk″ ; fk vk+1← lk″ ;
}
end for
vnv1 ← yn′ ; fnf1 ← hn′ ;
fng1 ← hn″; fn v1← ln″ ;
for k= 1 to n
{
gkfk← dk′ ; vkfk← lk′ ;
}
end for
v← n+1
for k=1 to n
{
ek← k ;
}
end for
for k=1 to n
{
wk← n+1 ;
}
end for
for k=1 to n
{
r← k+2 ;
if r ≤ n,
vk← r ;
else
vk← r-n ;
}
end for
for k=1 to n
{
s← k+4 ;
if s ≤ n,
fk← s ;
else
fk← s-n ;
}
end for
for k=1 to n
{
t← k+1 ;
if t ≤ n,
gk← t ;
else
gk← t-n;
}
end for
```

4.4 Theorem
For any Helm graph Hn,
a[T(Hn)]=n+1, n≥5.

Proof

As the two neighbors of wk (k=1 to n) have different colors, any bicolored cycle in T(Hn) can not contain any wk.

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Case(i)
Consider the colors (n+1) and k, (k=1 to n). As wk (k=1 to n) can not be contained in any bicolored cycle and then v is the only vertex with color n+1,T(Hn) is ((n+1),k)- cycle free.

Case(ii)
Consider the colors k and k+1, (k=1 to n-1). The induced subgraph of these color classes contains the bicolored path gk ek ek+1 fu fx vy vz, when n=5(where u=n+k-4,if k<4, u= k-4, if k ≥ 4, x=n+k-3, if k < 3 and x=k-3, if k ≥ 3,y=n+k-2, if k< 2 and y=k-2 ,if k ≥2, z=n+k-1,if k < 1 and z=k-1,if k ≥ 1)and the bicolored paths fu fx vyvz gz and gk ek ek+1 when n≥ 6. Thus, T(Hn) is (k,(n+1))- cycle free graph.

Case(iii)
Consider the colors j and k, 1≤ j,k≤ n and k≠j+1. The induced subgraph of these color classes is a linear forest as they contain only bicolored paths of various length (the paths varies with |j-k|). Therefore, T(Hn) is (j,k)-cycle free.

Thus, T(Hn) has no bicolored cycle in all the three cases and hence the coloring is acyclic. By (i) of sec 4.2, T(Hn) need minimum n+1 colors. Therefore, a[T(Hn)]= n+1, n≥5.

4.5 Remarks
(i) a[T(H3)]= 7.
(ii) a[T(H4)]= 7

Conclusion
We found the exact value of acyclic chromatic number of Middle, Central and Total graph of Helm graph families as follows:

(i) a[M(Hn)]= n+1, n≥7.
(ii) a[C(Hn)]= 2n-1, n≥4.
(iii) a[T(Hn)]= n+1, n≥5.

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