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Solutions of the Coupled Klein-Gordon Equation by Modified Exp-Function Method

KEYWORDS: Coupled Klein-Gordon Equation, exact wave solution, modified exp-function method

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ABSTRACT *The modified exp-function method is used to seek wave solutions of Coupled Klein-Gordon Equation. As a result, some new types of exact wave solutions are obtained which include periodic wave solution, and solitary wave solutions. Obtained results clearly indicate the reliability and efficiency of the proposed modified exp-function method.*

1. Introduction

The investigation of exact solutions of nonlinear wave equations plays an important role in the study of nonlinear physical phenomena. Recently, many effective methods for obtaining exact solutions of nonlinear wave equations have been proposed, such as bäcklund transformation method [1], homogeneous balance method [2,3], bifurcation method [4], Hirotas bilinear method [5], the hyperbolic tangent function expansion method [6,7], the Jacobi elliptic function expansion method [8,9], F-expansion method[10-12] and so on. He and Wu [13] developed the exp-function method to seek the solitary, periodic and compaction like solutions of nonlinear differential equations. Based on this method, modified exp-function expansion method is proposed. The purpose of this article is to find exact wave solutions of Klein-Gordon equation by the new method.

The nonlinear Coupled Klein-Gordon equation was first studied by Alagesan et al. (2004)[14] and then Shang (2010) [15]. Yusufoglu and Bekir (2008) [16] gave further results by using the tanh method and the general integral method and obtained the periodic solutions. In a paper of Sassaman and Biswas (2009) [17] the quasilinear Coupled Klein-Gordon, which have several forms of power law nonlinearity, are well studied by using soliton perturbation theory.

Considering the Coupled Klein-Gordon equation

$$E_{tt} - E_{x} + \mathrm{P}E + \alpha E^3 + \beta\eta E = 0 \qquad (1)$$
$$\eta_t - \eta_x + 4\mathrm{E}\ \ _t = 0$$

Where α, β and P are constants. The nonlinear Coupled Klein-Gordon equation appears in many types of nonlinearities and several methods are used to investigate these types of equations. The aim of this work is to further complement the studies on the Coupled Klein Gordon equations.

2. Modified exp-function method

The exp-function method was first proposed by He and Wu to solve differential equations [13] and it was systematically studied in [14-17]. The main procedures of this method are as follows. We consider a general nonlinear PDE in the form

$$H(E, E_t, E_x, E_t, E_x\ E_{\kappa}\ ....) = 0 \qquad (2)$$

Where *E=E* (x, t) is the solution of the Eq. (2). We use transformations

$$\xi = x - t \qquad (3)$$

Where c are constants, to obtain

$$\frac{\partial}{\partial t}(\ ) = -c\frac{\partial}{\partial \xi}(\ ), \frac{\partial}{\partial x}(\ ) = \frac{\partial}{\partial \xi}(\ ), \frac{\partial^2}{\partial t^2}(\ ) = c^2\frac{\partial^2}{\partial \xi^2}(\ )$$

We use (3) to change the nonlinear partial differential equation (1) to the nonlinear ordinary differential equation

$$H_1(E, E', E'', E''', ..) = 0 \qquad (4)$$

Where the prime denotes the derivation with respect to ξ. Let

$$E = v + s \qquad (5)$$

Where s is constants Then Eq. (4) becomes

$$H_2(v, v', v'', v'''.....) = 0 \qquad (6)$$

Assume that the solution of Eq. (4) can be expressed in the following form

$$E(\xi) = \frac{\sum_{i=-n}^{n} \chi_i g^i}{\sum_{i=-n}^{n} \varepsilon_i g^i} = \frac{\sum_{i=0}^{2n} b_i g^i}{\sum_{i=0}^{2n} a_i g^i} \qquad (7)$$

where $g = e^{-k\xi}$ which is the solution of the homogeneous linear equation corresponding to equation (6), a_i, b_i are unknown to be further determined and n can be determined by homogeneous balance principle. Substituting equation (7) into (4), we can get polynomial equation in g. If the coefficient of g^i be zero, and on solving the equation set, the a_i, b_i can be determined.

3. Solutions of the Coupled Klein-Gordon Equation

Using the wave transformations $E(x,t) = E(\xi)$, $\eta(x,t) = \eta(\xi)$, $\xi = x - t$ the equation (1) can be rewritten as

$$(c^2 - 1)E'' + \mathrm{P}E + \alpha E^3 + \beta\eta E = 0 \qquad (8)$$
$$(c+1)\ \eta' + 4cEE' = 0$$

Let $E = u + s$, $\eta = v + a$ Where s and a is constants Then Eq. (8) becomes

$$(c^2 - 1)u'' + \mathrm{P}(u+s) + \alpha(u+s)^3 + \beta(v+a)(u+s) = 0 \qquad (9)$$
$$(c+1)v' + 4c(u+s)u' = 0$$

By integrating the second equation with respect to ξ and neglecting the constant of integration we obtain

$$v(\xi) = -\frac{2c}{c+1}u^2(\xi) - \frac{2s}{c+1}u(\xi) \qquad (10)$$

Substituting (10) into the first equation of equation (9) and integrating the resulting equation, we find

$$(c^2 - 1)u'' + \mathrm{P}(u+s) + \alpha(u+s)^3 + \beta\left(-\frac{2c}{c+1}u^2 - \frac{2s}{c+1}u + a\right)(u+s) = 0 \qquad (11)$$

Let $s = \pm\sqrt{-\frac{(\mathrm{P} + a\beta)}{\alpha}}$, then $s = 0$ and $(\alpha s^3 + \mathrm{P}s + a\beta s) = 0$

Case 1: $s = 0$

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The solution of the linear equation corresponding to equation (11) is

$$g = e^{-k\xi} \quad k = \sqrt{\frac{\mathrm{P} + a\beta}{1 - c^2}} \tag{12}$$

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Thus, we look for the solution of (11) in the form

$$u(\xi) = \frac{\sum_{i=0}^{2n} b_i g^i}{\sum_{i=0}^{2n} a_i g^i} = \frac{b_0 + b_1 g + b_2 g^2}{a_0 + a_1 g + a_2 g^2} \tag{13}$$

Substituting Eq. (13) and Eq. (12) into Eq. (11) yields a set of algebraic equations for g^i (i = 0, 1,...6). If the coefficient of these terms g^i are zero then it yields a set of over-determined algebraic equation

$$\left(\alpha - \frac{2c\beta}{c+1}\right) b_0^{\ 2} + (\mathrm{P} + a\beta) b_0 a_0^2 = 0,$$

$$3(\mathrm{P} + a\beta) b_0 a_1 a_0 + 3\left(\alpha - \frac{2c\beta}{c+1}\right) b_0^2 b_1 = 0$$

$$3\left(\alpha - \frac{2c\beta}{c+1}\right) b_0^2 b_2 + 3\left(\alpha - \frac{2c\beta}{c+1}\right) b_1^2 b_0 - 3(\mathrm{P} + a\beta) b_2 a_0^2 + 3(\mathrm{P} + a\beta) b_1 a_1 a_0 + 6(\mathrm{P} + a\beta) b_0 a_2 a_0 = 0$$

$$(\mathrm{P} + a\beta) b_1 a_1^2 - (\mathrm{P} + a\beta) a_2 a_1 + 6\left(\alpha - \frac{2c\beta}{c+1}\right) b_0 b_1 b_2 + 8(\mathrm{P} + a\beta) b_1 a_0 a_2 - (\mathrm{P} + a\beta) b_2 a_1 a_0 + \left(\alpha - \frac{2c\beta}{c+1}\right) b_1^3 = 0$$

$$3(\mathrm{P} + a\beta) b_1 a_0 a_2 + 3\left(\alpha - \frac{2c\beta}{c+1}\right) b_1^2 b_2 - 3(\mathrm{P} + a\beta) b_0 a_2^2 + 6(\mathrm{P} + a\beta) b_2 a_2 a_0 + 3\left(\alpha - \frac{2c\beta}{c+1}\right) b_2^2 b_0 = 0$$

$$3\left(\alpha - \frac{2c\beta}{c+1}\right) b_2^2 b_1 + 3(\mathrm{P} + a\beta) b_2 a_1 a_2 = 0$$

$$(\mathrm{P} + a\beta) b_2 a_2^2 + \left(\alpha - \frac{2c\beta}{c+1}\right) b_2^{\ 3} = 0$$

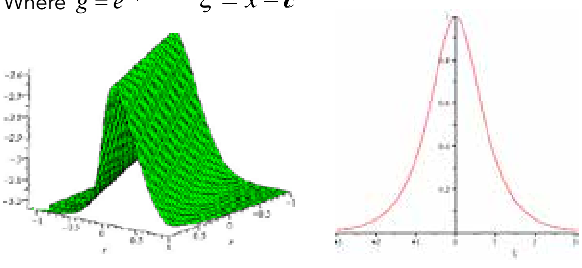
Solving the system of algebraic equations

$$a_0 = -\frac{\left(\alpha - \frac{2c\beta}{c+1}\right) b_1^2}{8(\mathrm{P} + a\beta) a_2} \quad a_1 = 0, b_0 = 0, b_2 = 0$$

Wher $\alpha_2 \neq 0, b_1$ is arbitrary constant, then E(x,t)

$$E(\xi) = u(\xi) = \frac{b_1 g}{\left(-\frac{\left(\alpha - \frac{2c\beta}{c+1}\right) b_1^2}{8(\mathrm{P} + a\beta) a_2} + a_2 g^2\right)} \quad v(\xi) = -\frac{2c}{c+1}\left(\frac{b_1 g}{-\frac{\left(\alpha - \frac{2c\beta}{c+1}\right) b_1^2}{8(\mathrm{P} + a\beta) a_2} + a_2 g^2}\right)^2 \tag{14}$$

Where $g = e^{-\sqrt{\frac{\mathrm{P} + a\beta}{1 - c^2}}\xi}$ $\xi = x - t$



(a) (b)

Fig1: 3-D and 2-D figures of Periodic Solution for equation (11) for α = -1, β=2 and c=2, s= 0

Case 2: $s = \pm\sqrt{-\frac{(\mathrm{P} + a\beta)}{\alpha}}$

The solution of the linear equation corresponding to equation (11) is

$$g = e^{-k\xi} \quad , \quad k = \sqrt{\frac{2(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)}\left(\alpha - \frac{2c\beta}{c+1}\right)} \tag{15}$$

Thus, we look for the solution of (11) in the form

$$E(\xi) = \frac{\sum_{i=0}^{2n} b_i g^i}{\sum_{i=0}^{2n} a_i g^i} = \frac{b_0 + b_1 g + b_2 g^2}{a_0 + a_1 g + a_2 g^2} \tag{16}$$

Substituting Eq. (15) and Eq. (16) into Eq. (11) yields a set of algebraic equations for g^i (i = 0, 1,...6). Let the coefficient of these terms g^i to be zero yields a set of over-determined algebraic equations.

$$b_0^{\ 3} + \frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_0 a_0^2 = 0,$$

$$3b_0^{\ 2} b_1 + 3\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} a_0^2 b_1 = 0$$

$$9\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} a_0^2 b_2 + 3b_0^2 b_2 + 3b_0 b_1^2 + 3\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} a_1^2 b_0 - 6\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_0 a_2 a_0 = 0$$

$$8\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} a_1 a_2 b_0 + \frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_1 a_1^2 + 6b_0 b_1 b_2 + 8\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_2 a_0 a_1 - 10\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_1 a_2 a_0 + b_1^3 = 0$$

$$3\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_2 a_1^2 + 3b_1^2 b_2 - 6\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_2 a_0 a_2 + 9\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_0 a_2^2 + 3b_2^2 b_0 = 0$$

$$3b_2^2 b_1 + 3\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_1 a_2^2 = 0$$

$$\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_2 a_2^2 + b_2^{\ 3} = 0$$

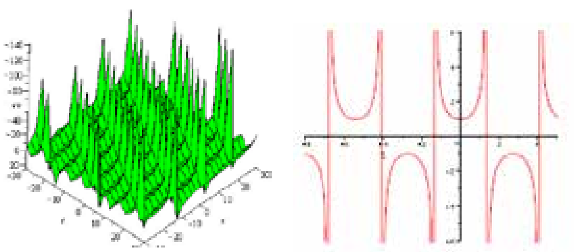
Solving the system of algebraic equations by use of Maple, we obtain

$$a_0 = \pm\frac{\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} + b_1^2}{4\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} b_2 \sqrt{-\frac{1}{\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)}}}} \quad a_2 = \pm b_2 \sqrt{-\frac{1}{\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)}}}, \quad b_0 = \frac{\frac{(\mathrm{P} + a\beta)}{\alpha(c^2 - 1)} a_0^2 + b_1^2}{4b_2}$$

Where $b_2 \neq 0, b_1$ a_1 is arbitrary constant, then

$$u(\xi) = \frac{\frac{(\mathrm{P} + a\beta) a_0^2 + \alpha(c^2 - 1) b_1^2}{4b_2 \alpha(c^2 - 1)} + b_1 g + b_2 g^2}{\pm\frac{(\mathrm{P} + a\beta) + \alpha(c^2 - 1) b_1^2}{4(\mathrm{P} + a\beta) b_2 \sqrt{-\frac{\alpha(c^2 - 1)}{(\mathrm{P} + a\beta)}}} + a_1 g \pm \sqrt{-\frac{\alpha(c^2 - 1)}{(\mathrm{P} + a\beta)}} b_2 g^2}$$

$$v(\xi) = -\frac{2c}{c+1} u^2(\xi) - \frac{2\varepsilon}{c+1} u(\xi)$$



(a) (b)

Fig2: 3-D and 2-D figures of Periodic Solution for equation (11) for α = -1, α=2 and c=1/2 ,

$s = \pm\sqrt{-\frac{(\mathrm{P} + a\beta)}{\alpha}}$

4. Conclusions

In this paper, we have obtained the solutions of the Coupled Klein- Gordon equation by the modified exp-function method. It shows that the new method is powerful and straightforward for nonlinear differential equations. It is said that this method can be applied to other kinds of nonlinear problems also.

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