Common Fixed Point Theorems for Sequence of Mappings Under Partial Metric Spaces

KEYWORDS: Common fixed point, coincidence point, weakly compatible pair of mappings, partial metric space

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ABSTRACT *The main purpose of this paper is to obtain fixed point theorems for sequence of mappings under partial metric spaces which generalizes theorem of four authors [5].*

1. INTRODUCTION

Partial metric spaces were introduced by Matthews [1] in 1992 as a part of the study of denotational semantics of dataflow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation.

2. PRELIMINARIES

Before proving our results we need the following definitions and known results in this sequel [1, 2, 4].

Definition2.1. ([1]). A partial metric on a nonempty set X is a function p: X × X ------>R+ such that for all x, y, z ε X:

(p1) x=y <==> p(x, x) = p(x, y) = p(y, y),

(p2) p(x, x) ≤ p(x, y),

(p3) p(x, y) = p(y, x),

(p4) p(x,y) ≤ p(x,z)+ p(z,y)-p(z,z).

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Remark 2.2. It is clear that, if p(x, y) = 0, then from (p1) and (p2), x = y. But if x = y, p(x, y) may not be 0. A basic example of a partial metric space is the pair (R+,p),

where p(x,y) = max{x,y} for all x; y ε R+ .Each partial metric p on X generates a T0 topology τp on X which has as a base the family of open p-balls {Bp(x, ε), x ε X, ε >0} where Bp(x, ε) = {y ε X: p(x, y) < p(x, x) + ε} for all x ε X and ε > 0.

If p is a partial metric on X, then the function ps : X × X ------>R+ given by

ps(x, y) =2p(x, y)-p(x, x)-p(y, y) is a metric on X.

Definition 2.3. Let (X, p) be a partial metric space and {xn} be a sequence in X. Then

(i){xn} converges to a point x ε X if and only if p(x, x) =

(ii) {xn} is called a Cauchy sequence if there exists (and is finite)

Definition 2.4. A partial metric space (X, p) is said to be complete if every Cauchy sequence {xn} in X converges, with respect to τp, to a point x ε X, such that

p(x, x) = $\lim_{n,m\to+\infty} p(x_n, x_m)$

Remark 2.5. It is easy to see that every closed subset of a complete partial metric space is complete.

Lemma 2.6 ([1, 2]). Let (X, p) be a partial metric space. Then (a) {xn} is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X,ps),

(b) (X, p) is complete if and only if the metric space (X,ps) is complete. Furthermore,

$$\lim_{n\to+\infty} p^s(x_n, x) = 0$$

if and only if

$$p(x, x) = \lim_{n\to+\infty} p(x_n x) = \lim_{n,m\to+\infty} p(x_n, x_m)$$

Matthews [1] obtained the following Banach fixed point theorem on complete partial metric spaces.

Theorem 2.7[1]. Let f be a mapping of a complete partial metric space (X, p) into itself such that there is a real number c with 0≤ c < 1, satisfying for all x, y ε X:p(fx,fy) ≤c p(x,y).

Then f has a unique fixed point.

3. MAIN RESULTS

Before stating the main results, we recall the following definitions.

Definition 3.1. Let X be a non-empty set and T_1, T_2: X→X are given self-maps on X.

If w = T_1x = T_2x for some x ε X, then x is called a coincidence point of T_1 and T_2, and w is called a point of coincidence of T_1 and T_2.

Definition 3.2 [3]. Let X be a non-empty set and T_1, T_2: X→X are given self-maps on X. The pair {T_1, T_2} is said to be weakly compatible if $T_1T_2t = T_2T_1t$, whenever $T_1t = T_2t$ for some t in X.

Our main result is the following.

Theorem 3.3. Suppose that Ai, Aj (i≠j) S, T are self-maps of a complete partial metric space (X, p) such that AiX ⊆ TX, AjX ⊆ SX (i≠j) and P (Aix; Ajy) ≤Φ (M(x, y)) ---- (3.1) for all

x, y ε X, where Φ ε Φ and M(x,y)=max{p(Sx,Ty),p(Aix,Sx), p(Ajy,Ty), 1/2[p (Sx, Ajy) +

p (Aix, Ty)]}.If one of the ranges AiX, AjX, TX and SX is a closed subset of (X, p), then (i) Ai and S have a coincidence point, (i≠j) (ii) Aj and T have a coincidence point. Moreover, if the pairs {Ai, S} and {Aj, T} (i≠j) are weakly compatible, then

Volume : 3 | Issue : 9 | Sept 2013 | ISSN - 2249-555X

Ai, Aj(i≠j), T and S have a unique common fixed point.

Proof. Let x0 be an arbitrary point in X. Since AiX ⊆TX, there exists x1 ε X such that

Tx1 = Aix0. Since AjX ⊆ SX, there exists x2 ε X such that Sx2 = Ajx1(i≠j). Continuing this process, we can construct sequences {xn} and {yn} in X defined by

y2n = Tx2n+1= Aix2n , y2n+1 = Sx2n+2= Ajx2n+1 ----- (3.2) for every n ε N (i≠j)

We claim that {yn} is a Cauchy sequence in the partial metric space (X, p).

We have: M (x2p, x2p+1) = max {p (Sx2p, Tx2p+1), p (Aix2p, Sx2p), p (Ajx2p+1,Tx2p+1),

½[p (Sx2p, Ajx2p+1) + P (Aix2p, Tx2p+1)]} for (i≠j),

M (x2p, x2p+1) = max {p (y2p-1, y2p), p (y2p, y2p-1), p (y2p+1,y2p), ½[p (y2p-1, y2p+1) + p (y2p, y2p)]}

M (x2p, x2p+1) = max {p (y2p-1, y2p), p (y2p+1, y2p), ½[p (y2p-1, y2p) + p (y2p, y2p+1)]}

M (x2p, x2p+1) = max {p (y2p-1, y2p), p (y2p+1, y2p)}

Since p (y2p-1, y2p+1) +p (y2p, y2p) ≤ p (y2p-1, y2p) +p (y2p, y2p+1).

Using that Φis non-decreasing function, we get:

Φ (M (x2p, x2p+1)) ≤ Φ (max {p (y2p-1, y2p), p (y2p, y2p+1)})

From the contraction condition (3.1) with x = x2p and y = x2p+1, we get:

p (y2p, y2p+1) ≤ Φ (max {p (y2p-1, y2p), p (y2p, y2p+1)}) ----- (3.3)

Similarly we obtain p (y2p+1, y2p+2) ≤ Φ (max {p(y2p, y2p+1),p (y2p+1, y2p+2) })-----(3.4)

Therefore, from (3.3) and (3.4),

p (yn, yn+1) ≤ Φ (max {p (yn-1, yn), p (yn, yn+1)}) for sufficiently large n------- (3.5)

Suppose that there exists p ε N such that p (y2p-1 y2p) = 0. Then we have y2p-1= y2p and from (3.3), we obtain: p (y2p, y2p+1) ≤ Φ ((p (y2p, y2p+1)).

Since Φ (t) < t for each t > 0, the above inequality implies that p (y2p, y2p+1) = 0 and then

y2p = y2p+1. From (3.4), we get: p (y2p+1, y2p+2) ≤ Φ (p (y2p+1, y2p+2)), which implies that

y2p+1= y2p+2. Hence, we have y2p-1 = y2p= y2p+1 = y2p+2 = ----------------

Then {yn} is a Cauchy sequence in (X, p). The same conclusion holds if we suppose that there exists p ε N such that p (y2p, y2p+1) = 0,

Now, we assume that p (yn, yn+1) > 0, for sufficiently large n ------ (3.6)

Then from (3.5), as Φ (t) < t for all t > 0, we have

p (yn, yn+1) < max {p (yn-1,yn), p (yn,yn+1)}.

Hence we get p (yn, yn+1) < (p(yn-1,yn).

Therefore, Max {p (yn-1,yn), p (yn,yn+1)} = p (yn-1, yn) for sufficiently large n

Thus from (3.5), p (yn, yn+1) ≤ Φ (p (yn-1,yn)) for sufficiently large n----- (3.7)

Repeating this inequality n time we obtain

p (yn, yn+1) ≤ Φn (p (y0,y1)) ------- (3.8)

By the properties (p2) and (p3) we have

Max {p (yn,yn), p (yn+1,yn+1)} ≤ p (yn, yn+1)

Thus from (3.8), max {p (yn,yn), p (yn+1,yn+1)} ≤ Φn p (y0, y1)------ (3.9)

Therefore,ps((yn,yn+1)=2p(yn,yn+1)- p(yn,yn)-p(yn+1,yn+1) ≤2p(yn,yn+1)+p(yn,yn)+p(yn+1,yn+1)

≤4Φn p ((y0, y1)).

Now by the triangle inequality for the metric ps and (3.9), for any k, n ε N* we have

ps(yn,yn+k) ≤ ps(yn,yn+1)+ ps(yn+1,yn+2)+ ----------------+ps(yn+k-1,yn+k),

≤ 4 Φn p ((y0, y1)) + 4Φn+1 p ((y0, y1)) +-------------+4Φn+k-1 p ((y0, y1))

$$\leq 4\,(\textstyle\sum_{i=n}^{n+k-1} \Phi^i(p\,(y_0,y_1)))$$

$$\leq 4\,(\textstyle\sum_{i=n}^{\infty} \Phi^i(p\,(y_0,y_1)))$$

Hence and from the property (b) of Φ we conclude that for an arbitrary ε > 0 there is a positive integer n0 such that ps(yn,yn+k) < ε for every n≥ n0 and all k ε N

Thus we proved that {yn} is a Cauchy sequence in the metric space (X,ps).

Since (X, p) is complete, then from Lemma2.6, (X,ps) is a complete metric space.

Therefore, the sequence {yn} converges to some y ε X, that is,

From the properties (b) in Lemma 2.6, we have

$$p(y,y) = \lim_{n\to+\infty} p(y_n,y) = \lim_{m\geq n\to+\infty} p(y_n,y_m) \text{- } (3.10)$$

Moreover, since {yn} is a Cauchy sequence in the metric space (X,ps), then

$$\lim_{n,m\to+\infty} p^s(y_n,y_m) = 0$$

and so from (3.9) and the property (b) of Φ we have -----------

$$\lim_{n\to+\infty} p(y_n,y_n) = 0\text{--} \qquad (3.11)$$

Thus from the definition of ps and (3.11), we have

$$\lim_{m\geq n\to+\infty} p(y_n,y_m) = 0$$

Therefore, from (3.10), we have

$$p(y,y) = \lim_{n\to+\infty} p(y_n,y) = \lim_{m\geq n\to+\infty} p(y_n,y_m) = 0 \text{ ----------(3.12)}$$

This implies that $\lim_{n\to+\infty} p(y_{2n},y) = \lim_{n\to+\infty} p(y_{2n-1},y) = 0$ ----------- (3.13)

Thus from (3.13) we have

$\lim_{n\to+\infty} p(A_i x_{2n}, y) = \lim_{n\to+\infty} p(Tx_{2n+1}, y) = 0$ ------- (3.14)

and $\lim_{n\to+\infty} p(A_j x_{2n-1}, y) = \lim_{n\to+\infty} p(Sx_{2n}, y) = 0$ --- (3.15)

Now we can suppose, without loss of generality, that SX is a closed subset of the partial metric space (X, p). From (3.15), there exists u ε X such that y = Su. We claim that

p (Aiu, y) =0. Suppose, to the contrary, that p (Aiu, y) > 0.

By (p4) and (3.1) we get

p (y,Aiu) ≤ p (y,Ajx2n+1) +p (Aiu,Ajx2n+1) - p (Ajx2n+1, Ajx2n+1) for (i≠j)

≤ p (y,Ajx2n+1) +p (Aiu,Ajx2n+1)

≤ p (y,Ajx2n+1)+ Φ (M(u,x2n+1))

By (3.2) we have M(u,x2n+1)=max{p(y,y2n)p(Aiu,y), p(y2n+1,y 2n),1/2[p(y,y2n+1)+p(Aiu,y2n)] }

≤ max{p(y,y2n)p(Aiu,y), p(y2n+1,y2n),1/2[p(y,y2n+1)+p(Aiu,y) +p(y,y2n),p(y,y)] }------------(3.16)

Since Φ is continuous, from (3.16), (3.12), and letting n→∞ we obtain

$p(y, A_i u) \leq \lim_{n\to\infty}[p(y_n, y_{2n+1}) + \Phi(M(u, x_{2n+1}))]$

$= \lim_{n\to\infty} p(y_n, y_{2n+1}) + \Phi(\lim_{n\to\infty} M(u, x_{2n+1}))$

$= \Phi(p(A_i u, y))$.

Hence, as we supposed that p (Au, y) > 0 and as Φ (t) < t for t > 0, we have

P(y, Aiu) < p(y, Aiu) which is a contradiction.

Therefore, p(Aiu,y) = 0

=>y=Aiu----------------------- (3.17)

Since y = Su, then Aiu = Su, that is, u is a coincidence point of Ai and S.

Hence the proof of (i).

Since AiX ⊆ TX and (3.17), we have y ε TX.

Therefore there exists v ε X such that y = Tv. We claim that

p (Ajv, y) = 0. Suppose, to the contrary, that p (Ajv, y) > 0. From (3.1) we have

p(y,Ajv)=p(Aiu,Ajv)≤ Φ (M(u,v))-----------(3.18)

where M(u,v)=max{p(Su,Tv),p(Aiu,Su), p(Ajv,Tv),1/2[p(Su, Ajv)+p(Aiu,Tv)] }.

= max{p(y,y),p(y,y), p(Ajv,y),1/2[p(y, Ajv)+p(y,y)] }.(by 3.17)

Here y = Su = Aiu = Tv. Hence by (3.12),

M (u, v) =p (Ajv, y)

Thus from (3.18), we have p (Ajv, y) ≤ Φ (p (Ajv, y)) < p (Ajv, y)

This is a contradiction. Then, we deduce that p (Ajv, y) =0 and y=Ajv=Tv-------- (3.19)

Therefore v is a coincidence point of Aj and T, then (ii) holds.

Since the pair {Ai, S} is weakly compatible, from (3.17), we have Aiy = AiSu = SAiu = Sy.

We claim that p (Aiy, y) =0. Suppose, to the contrary, that p (Aiy, y) > 0. We have

p(Aiy,y)≤ p(Aiy,y2n+1))+p(y2n+1,y)

= (p (Aiy,Ajx2n+1)) +p (y2n+1, y)

≤Φ (M(y,x2n+1)) +p (y2n+1, y) --------------- (3.20)

On the other hand, we have

M(y,x2n+1)= max{p(Sy,Tx2n+1),p(Aiy,Sy), p(Ajx2n+1,Tx2n+1),1/2[p(Sy, Ajx2n+1)+p(Aiy,T_2n+1)] }.

= max{p(Aiy,y2n),p(Aiy, Aiy), p(y2n+1,y2n),1/2[p(Aiy,y2n+1)+p(Aiy,y2n)] }.

Using (3.12) and (p2), we get

M(y,x2n+1)= max{p(Aiy,y), p(Aiy, Aiy),0,p(Aiy,y)}

= p (Aiy, y) as n→+∞ ---------------- (3.21)

Using (3.21), the continuity of Φ, (3.12) and letting n + ∞ in (3.20), we obtain

p (Ajy, y) ≤ Φ (p (Aiy, y)) < p (Aiy, y),

Which is a contradiction.

Then we deduce that p (Aiy, y) =0 and Aiy=Sy=y------------- (3.22)

Since the pair {Aj, T} is weakly compatible, from (3.19), we have Ajy = AjTv = T Ajv = Ty. We claim that p (Ajy, y) = 0.

Suppose, to the contrary, that p (Ajy, y) > 0, then by (3.1) and (3.22), we have

p(y,Ajy)= p(Aiy,Ajy)≤Φ (M(y,y)), where M(y,y)=max{p(Sy,Ty),p(Aiy,Sy),p(Ajy,Ty),

1/2[p (Sy, Ajy) +p (Aiy, Ty)]}.

=max{p(y,Ajy),p(y,y), p(Ajy,Ajy),1/2[p(y, Ajy)+p(y,Ajy)] }.

= p (Ajy, y).

Thus, we get p(y,Ajy) ≤Φ (p (Ajy, y))) <p (Ajy, y),

This is a contradiction. We deduce that

p(y,Ajy) =0 and Ajy=Ty=y--------- (3.23)

Now, combining (3.22) and (3.23), we obtain

y=Aiy=Ajy=Sy=Ty, (i≠j).

That is, y is a common fixed point of Ai, Aj, S and T.

Uniqueness

Let us suppose that z ε X is a common fixed point of Ai, Aj, S and T, with p (z, y) > 0.

Using (3.1), we get p (y,z)=p(Aiy,Ajz)

≤ Φ (max{p (Aiy,Ajz),p (Aiy,Ajy),p(Ajz,Ajz),1/2[p(Aiy, Ajz) +p(Aiz, Ajy)]})

= Φ (max{p(y,z),p(y,y)p(z,z)})=Φ(p(y,z))<p(y,z)

Which is a contradiction. Then we deduce that z = y.

Therefore, the uniqueness of the common fixed point is proved.

That is, the proof of the theorem is complete.

Corollary 3.4[5]. Suppose that A, B, S, and T are self-maps of a complete partial metric space (X, p) such that AX $\subseteq$ TX, BX $\subseteq$ SX and p(Ax, By) ≤Φ (M(x, y)) for all x, y ε X, where Φ ε Φ and M(x,y)=max{p(Sx,Ty),p(Ax,Sx), p(By,Ty),1/2[p(Sx,By)+p(Ax,Ty)] }.

If one of the ranges AX, BX, TX and SX is a closed subset of (X, p), then (i) A and S have a coincidence point, (ii) B and T have a coincidence point.

Moreover, if the pairs {A, S} and {B, T} are weakly compatible, then A, B, T and S have a unique common fixed point.

Corollary 3.5. Suppose that S and T are self-maps of a complete partial metric space (X, p) such that TX $\subseteq$ SX and p(TX, Ty) ≤Φ (M(x, y)) for all x, y ε X, where Φ ε Φ and M(x,y)≤max{p(Sx,Sy),1/2[p(Tx,Sx)+p(Ty,Sy)],1/2[p(Ty,Sx)+p(Tx,Sy)] }.

If one of the ranges TX and SX is a closed subset of (X, p), then (i) S and T have a coincidence point, (ii) Moreover, if the pairs { S,T} is weakly compatible, then T and S have a unique common fixed point.

Proof. The proof follows from above theorem 3.3.

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