Domination of Complement of A Splitted Graphs

KEYWORDS
Domination, Splitted graph, complement of splitted graphs

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ABSTRACT
Let G be \((p,q)\) graph. A set \(D\) of vertices in a complement of any splitted graph \([S(G)]^C\) is called a dominating set if every vertex in \(V-D\) is adjacent to some vertex in \(D\). The domination number \(\gamma([S(G)]^C)\) of \([S(G)]^C\) is minimum cardinality of a domination set of \([S(G)]^C\).

1. INTRODUCTION
Let G be a \((p,q)\) graph. By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G denoted are by \(V(G)\) and \(E(G)\) respectively. Let G be a graph. For each vertex \(v\) of a graph \(G\), take a new vertex \(u\). Join \(u\) to those vertices of \(G\) adjacent to \(v\). The graph thus obtained is called the splitting graph of \(G\). It is denoted by \(S(G)\).

2. PRELIMINARIES
Let G be \((p,q)\) graph. A set \(D\) of vertices in a complement of any splitted graph \([S(G)]^C\) is called a dominating set if every vertex in \(V-D\) is adjacent to some vertex in \(D\). The domination number \(\gamma([S(G)]^C)\) of \([S(G)]^C\) is minimum cardinality of a domination set of \([S(G)]^C\).

3. RESULTS
Theorem 3.1. Let \([S(G)]^C\) be a Complement of splitted graph. If \(D\) is a minimal domination set, then \(V-D\) is also a dominating set.

Proof. Let \(S(G)\) be a splitted graph. Let \([S(G)]^C\) be a Complement of \(S(G)\). Let \(D\) be a minimal dominating set of \([S(G)]^C\). Suppose \(V-D\) is not a dominating set. Then there exists a vertex \(u\) such that \(u\) is not dominated by any vertex in \(V-D\). \(u\) is dominated by atleast one vertex in \(D-\{u\}\), which is contradicts the minimality of \(D\). Thus every vertex in \(V\) is adjacent with atleast one vertex in \(V-D\). Hence \(V-D\) is a dominating set.

For example complement of \(S(P_2)\) is shown in the following figure 3.2 and figure 3.3 respectively

Result 3.4 Complement of all complete graph is disconnected, It does not have a dominating set.

Result 3.5. Complement of all complete bipartite graph is disconnected, It does not have a dominating set.

Result 3.6. Complement of a cycle \(c_n\) \((n < 4)\) is disconnected, It does not have a dominating set.

Result 3.7. Complement of a cycle \(c_n\) \((n \geq 5)\) is connected graph, It has a dominating set.

Result 3.8. Complement of path is disconnected, It does not have a dominating set.

Theorem 3.9. For any splitted graph \(G\), \(\gamma([S(G)]^C) = 2\).
Proof. Let $S(G)$ be splitted graph of a graph $G$. Let $[S(G)]^C$ be complement of $S(G)$. Let $D$ be a minimal dominating set of $[S(G)]^C$. $\text{V-Dis}$ a domination set of $[S(G)]^C$.

Hence, $|V - \text{N}(v)| = \delta$. Hence $V - \text{N}(v)$ is a dominating set.

Theorem 3.10. $\gamma$ $[S(G)]^C = P - \Delta [S(G)]^C$

Proof. Let $S(G)$ be splitted graph of a graph $G$. Let $[S(G)]^C$ be complement of $S(G)$. Let $v$ be a vertex of maximum degree in $S(G)$. Then $v$ is adjacent to $\text{N}(v)$ vertices such that $\Delta [S(G)]^C = N(v)$. Hence $V - \text{N}(v)$ is a dominating set.

Thus $\gamma$ $[S(G)]^C = |V - \text{N}(v)| = P - \Delta [S(G)]^C$

Theorem 3.11. $\delta [S(G)]^C = P - (\Delta [S(G)] + 1)$

Proof. Let $S(G)$ be splitted graph of a graph $G$. Let $[S(G)]^C$ be complement of $S(G)$. Let $v$ be a vertex of maximum degree in $S(G)$. Then $v$ is adjacent to $\text{N}(v)$ vertices, such that $\Delta [S(G)]^C = \text{deg}(v)$. It implies $\delta$ is the vertex of minimum degree in $[S(G)]^C$.

Then $\delta [S(G)]^C = P - (\Delta [S(G)] + 1)$

Theorem 3.12. $\Delta [S(G)]^C = P - (\delta [S(G)] + 1)$

Proof. Let $S(G)$ be splitted graph of a graph $G$. Let $[S(G)]^C$ be complement of $S(G)$. Let $v$ be a vertex of minimum degree in $S(G)$. Then $v$ is adjacent to $\text{N}(v)$ vertices such that $\delta [S(G)]^C = \text{deg}(v)$. It implies $v$ is a vertex of maximum degree in $[S(G)]^C$ of complement of splitted graph.

Then $\Delta [S(G)]^C = P - (\delta [S(G)] + 1)$

Observation 3.13. $\gamma [S(G)]^C \leq \{P + 1 - (\delta [S(G)]^C) - 1\} \Delta [S(G)]^C / 2$

Observation 3.14. $\gamma [S(G)]^C \leq \{P + 2 - (\delta [S(G)]^C) / 2$

Observation 3.15. $\gamma [S(G)]^C \leq \{P + 1 - \delta [S(G)]^C\} / 2$

Observation 3.16. $q < \frac{1}{2} (P - \gamma [S(G)]^C)(P - \gamma [S(G)]^C + 2)$

Theorem 3.17. $\gamma [S(G)]^C < P + 1 - \sqrt{1 + 2q}$

Proof. By observation 3.16, $2q < (P - \gamma [S(G)]^C)^2 + 2(P - \gamma [S(G)]^C)$

Adding 1 to both sides $2q + 1 < (P - \gamma [S(G)]^C)^2 + 2(P - \gamma [S(G)]^C) + 1$

$\Rightarrow \gamma [S(G)]^C < P + 1 - \sqrt{1 + 2q}$

Observation 3.18. $S(P_2) \cong S(P_2)^C$

For example complement of $S(P_2)$ is shown in the following figure 3.19 and figure 3.20

Define 3.21. isomorphism

we define $\theta : V(S(P_2) \to V(S(P_2)^C)$
by \((v_1, u_2, u_1, v_2)\) and
\[
\begin{pmatrix}
v_1^1 \\ v_2^1 \\ v_1^2 \\ v_2^2 
\end{pmatrix}
\]
we define \(\emptyset : E(S(P_2)) \to E(S(P_2)^c)\)
by \((e_1, e_2, e_3)\)
\[
\begin{align*}
\varphi_{S(G)}(e_1) &= v_1 u_2, & \varphi_{S(G)^c}(\emptyset (e_1)) &= v_1^1 v_2^1 = \theta(v_1) \theta(u_2) \\
\varphi_{S(G)}(e_2) &= u_2 u_1, & \varphi_{S(G)^c}(\emptyset (e_2)) &= u_2^1 u_1^1 = \theta(u_1) \theta(u_2)
\end{align*}
\]
\[
\begin{align*}
\varphi_{S(G)^c}(e_2) &= v_1^1 v_2^1 = \theta(u_2) \theta(v_1) \\
\varphi_{S(G)^c}(e_3) &= v_2^1 u_2^1 = \theta(u_2) \theta(v_2)
\end{align*}
\]
\((\theta, \emptyset)\) is an isomorphism
\(S(P_2) \cong S(P_2)^c\)

**Note 3.22.** No other graph example will satisfy the above isomorphism observation 3.18.