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Analytical Expressions for the Concentration of Substrate, Prey and Predator in Chemostat Model

KEYWORDS: chemostat, prey, predator, mathematical modeling, Homotopy perturbation method

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ABSTRACT *Mathematical model of predator-prey interaction in a chemostat is discussed. In this paper, a new approach to Homotopy perturbation method is used to solve the non-linear differential equations in predator-prey systems. Using this method, the analytical expressions for the concentration of substrate, prey and predator have been obtained. An agreement between analytical and numerical results is noted.*

INTRODUCTION

A chemostat is a bioreactor to which fresh medium is continuously added, while culture liquid is continuously removed to keep the culture volume constant [1, 2]. By changing the rate with which medium is added to the bioreactor the growth rate of the microorganism can be easily controlled. Chemostat is a standard example of an open system with purely exploitative competition in a laboratory device [3]. The chemostat is perhaps the best laboratory idealization of nature for population studies [4]. In research, chemostat are used for investigations in cell biology, as a resource for large volumes of the same cells or protein. It is often used to gather steady state data about an organism in order to generate a mathematical model relating to its metabolic processes. In the traditional chemostat equations two "constants" are under the control of the experimental, the concentration of the input nutrient and the overflow rate [5].

The input and removal of nutrients to and from the chemostat represent the continuous cut over of nutrients in nature. Modeling is based on compartmental analysis, laws of mass exploit, and mass equilibrium. The dynamics of predator prey and substrate interaction in chemostat model has become important tool for studying a number of industrial fields such as waste treatment bioreactors [6]. An analysis is given for a mathematical model of two predators feeding on a single prey growing in the chemostat [7]. A mathematic model of such systems, featuring the common Michaelis-Menten kinetics of the uptake of the restrictive substrate, goes back to Monod [8].

Mehbuba Rehim et al. [9] discussed study a Monod type chemostat model with nutrient recycling and impulsive input in a polluted environment. B.W.Kooi et al. [10] study a predator-prey system is modeled by two ordinary differential equations which describe the rate of changes of the biomasses. Farsana Nasrin et al. [3] measured a model describing predator prey interaction in a chemostat that incorporates general response functions and identical removal rates.

Recently, Emad Ali et al. [11] developed a dynamical Chemostat model to analyze the interactions between one predator and prey in a chemostat. To the best of our knowledge, no rigorous analytical solutions of the model have been reported. The purpose of this communication is to provide the approximate analytical solutions for the concentrations substrate, prey, and predator for all values of the parameters. This analytical result will then be used to optimize other models of chaotic behavior in predator-prey interactions in a chemostat.

MATHEMATICAL MODELLING:

In chemostat model the mass balance equations for substrate S, prey X_1, and predator X_2 can be written as follows [11]:

$$\frac{dS(t)}{dt} = D(S_f - S(t)) - \frac{1}{Y_S(S)}\mu(S)X_1(t) \qquad (1)$$

$$\frac{dX_1(t)}{dt} = -DX_1(t) + \mu(S)X_1(t) - \frac{1}{Y_P}p(X_1)X_2(t) \qquad (2)$$

$$\frac{dX_2(t)}{dt} = -DX_2(t) + p(X_1)X_2(t) \qquad (3)$$

Where S_f is the substrate feed concentration, D is the dilution rate and $Y_S(S)$ and Y_P are the yield coefficients for prey growth on substrate and predator growth on prey, respectively. The yield Y_S is understood to depend on S while Y_P is implicit to be constant. $\mu(S)$ and $p(X_1)$ are the expressions for the specific growth rates for prey and predator, respectively. The specific growth rates are assumed to be in the following Monod-like form while the yield coefficient Y_S is assumed to be linear in S

$$\mu(S) = \frac{\mu_S S}{K_S + S}, p(X_1) = \frac{\mu_P X_1}{K_X + X_1}, Y_S(S) = a + bS \qquad (4)$$

μ_S and μ_P are maximum specific growth rates of prey and predator, respectively. K_S and K_X are saturation constants associated with the growth rates of prey and predator, respectively, a and b are constants describing the dependence of the yield coefficient Y_S on the substrate. Using the following variables [11],

$$\bar{S}(\bar{t}) = \frac{S(t)}{K_S}, \bar{S}_f = \frac{S_f}{K_S}, \bar{\mu} = \frac{\mu}{\mu_S}, \bar{D} = \frac{D}{\mu_S}, \bar{p} = \frac{p}{\mu_S}$$
$$\bar{X}_2(\bar{t}) = \frac{X_2(t)}{Y_S(K_S)K_S Y_P},, \bar{t} = t\mu_S, \bar{X}_1(\bar{t}) = \frac{X_1(t)}{Y_S(K_S)K_S} \qquad (5)$$

we get the dimensionless non-linear equations as follows:

$$\frac{d\bar{S}(\bar{t})}{d\bar{t}} = \bar{D}(\bar{S}_f - \bar{S}(\bar{t})) - \frac{\bar{\mu}(S)}{\bar{Y}(S)}\bar{X}_1(\bar{t}) \qquad (6)$$

$$\frac{d\bar{X}_1(\bar{t})}{d\bar{t}} = -\bar{D}\bar{X}_1(\bar{t}) + \bar{\mu}(\bar{S})\bar{X}_1(\bar{t}) - \bar{p}(\bar{X}_1)\bar{X}_2(\bar{t}) \qquad (7)$$

$$\frac{d\bar{X}_2(\bar{t})}{d\bar{t}} = -\bar{D}\bar{X}_2(\bar{t}) + \bar{p}(\bar{X}_1)\bar{X}_2(\bar{t}) \qquad (8)$$

when

$$\bar{\mu} = \frac{\bar{S}(\bar{t})}{1+\bar{S}(\bar{t})}, \bar{p} = \frac{\lambda\bar{X}_1(\bar{t})}{\bar{K} + \bar{X}_1(\bar{t})}, \bar{Y}(\bar{S}) = \frac{1+\lambda_1\bar{S}(\bar{t})}{1+\lambda_1}$$
$$\lambda = \frac{\mu_P}{\mu_S}, \bar{K} = \frac{K_X}{Y_S(K_S)K_S}, \lambda_1 = \frac{bK_S}{a} \qquad (9)$$

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The initial conditions are represented as follows:

At $\bar{t}=0,\ \bar{S}(0)=\bar{S}_0,\ \bar{X}_1(0)=\bar{X}_{10},\ \bar{X}_2(0)=\bar{X}_{20}$ (10)

When $\bar{t}\to\infty$, the steady state expressions of the concentrations for this model are obtained as follows:

$$\bar{S}=\bar{S}_f,\ \bar{X}_1=\bar{X}_2=0 \tag{11}$$

NEW APPROACH TO HOMOTOPY PERTURBATION METHOD:

The Homotopy perturbation method [12] was first proposed by Ji-Huan He in 1998 to solve nonlinear wave equations. The method employs a Homotopy transform to generate a convergent series solution of differential equations. The advantage of the method is that it does not need a small parameter in the system, leading to wide application in nonlinear wave equation [13]. Generally, the perturbation solutions are regularly valid as long as scientific system parameter is small. Recently, many authors have applied HPM to various problems and demonstrated the efficiency of the HPM for handling nonlinear engineering problems [14-18]. Recently a new approach to HPM is introduced to solve the nonlinear problem, in which we will get the better simple approximate solution in the zeroth iteration [19].

APPROXIMATE ANALYTICAL SOLUTION OF THE EQNS. (6) TO (8) USING NEW APPROACH TO HOMOTOPY PERTURBATION METHOD

In this paper, a new approach to Homotopy perturbation method is used (Appendix A) to solve the equations (1) to (3). Using this method, we have obtained the analytical expressions of concentration of substrate $\bar{S}$, prey $\bar{X}_1$ and predator $\bar{X}_2$ as follows:

$$\bar{S}(t)=\frac{\bar{D}\bar{S}_f}{A}+\left(\bar{S}_0-\frac{\bar{D}\bar{S}_f}{A}\right)e^{-At} \tag{12}$$

$$\bar{X}_1(t)=\frac{\bar{X}_{10}}{1+\bar{S}_0}\left(\frac{a}{A_1}+\frac{be^{-At}}{A_1-A}\right)+e^{-A_1t}\left(\bar{X}_{10}-\frac{\bar{X}_{10}A_2}{1+\bar{S}_0}\right) \tag{13}$$

$$\bar{X}_2(t)=\frac{\lambda\bar{X}_{20}}{K+\bar{X}_{10}}\left[\frac{\bar{X}_{10}\left(\frac{a}{\bar{D}A_1}+\frac{be^{-At}}{(A_1-A)(\bar{D}-A)}\right)}{1+\bar{S}_0}+\frac{e^{-At}\left(\bar{X}_{10}-\frac{\bar{X}_{10}A_2}{1+\bar{S}_0}\right)}{\bar{D}-A_1}\right]+\frac{\lambda\bar{X}_{20}}{K+\bar{X}_{10}}\left[\frac{\bar{X}_{10}\left(\frac{a}{\bar{D}-A_1}+\frac{b}{(A_1-A)(\bar{D}-A)}\right)}{1+\bar{S}_0}+\frac{\left(\bar{X}_{10}-\frac{\bar{X}_{10}A_2}{1+\bar{S}_0}\right)}{\bar{D}-A_1}\right] \tag{14}$$

Where,

$$A=\left(\bar{D}+\frac{\bar{S}_0\bar{X}_{10}(1+\lambda_1)}{(1+\bar{S}_0)(1+\lambda_1\bar{S}_0)}\right),\quad a=\frac{\bar{D}\bar{S}_f}{A},$$

$$b=\bar{S}_0-\frac{\bar{D}\bar{S}_f}{A},\quad A_1=\bar{D}+\frac{\lambda\bar{X}_{20}}{K+\bar{X}_{10}} \tag{15}$$

$$A_2=\frac{a}{A_1}+\frac{b}{A_1-A}$$

NUMERICAL SIMULATION

In order to investigate the accuracy of the new approach to HPM solution with a finite number of terms, the system of differential equations were solved numerically. To show the efficiency of the present method, our results are compared with numerical results graphically. The function ode45 (Range-Kutta method) in Matlab software [20, 21] which is a function of solving the initial value problems is used to solve Eqns. (1) to (3).

DISCUSSION

Eqns. (12) - (14) represents the simple approximate analytical expression for the concentrations a substrate, prey and predator respectively. Fig.1 shows the dimensionless concentration of substrate versus dimensionless time t for various values of $\bar{S}_f$. The concentration $\bar{S}$ increases when $\bar{S}_f$ increase and finally it reaches the steady state value $\bar{S}_f$. In Fig.2, we present the concentration of substrate $\bar{S}$ for various values of $\bar{D}$. From this figure, it is evident that when time increases the concentration decreases and finally it reaches the steady state value $\bar{S}_f=0.1$. The prey $\bar{X}_1$ versus time $\bar{t}$ is shown in Fig .3. The concentration $\bar{X}_1$ is independent of the parameter λ_1.

The profile of prey versus time for various values of λ is depicted in Fig.4. The concentration of prey increases when λ increases and it reaches the steady state value when $t\geq 0$ for all values of λ. The change in dimensionless concentration of predator $\bar{X}_2$ versus time t for the various values of λ is shown in Fig 5. The increase in concentration of predator $\bar{X}_2$ is due to the increase in λ. In Fig 6, the predator concentration $\bar{X}_2$ versus time t for various experimental values of dilution rate $\bar{D}$ is plotted. From this figure it is observed that, the concentration of predator decreases when dilution rate decreases. The concentration of substrate $\bar{S}$, prey $\bar{X}_1$, and predator $\bar{X}_2$ are plotted together for particular values of the parameters in Fig. 7.

CONCLUSION

A system of time dependent nonlinear differential equations in chemostat model has been solved analytically. A simple, closed- form analytical expression for the concentration of substrate $\bar{S}$, prey $\bar{X}_1$, and predator $\bar{X}_2$ are derived using new approach to HPM. The obtained analytical result can be easily extended to all kinds of system of non-linear ordinary differential equations in all fields like population growth model, epidemic model etc. Our results are compared with the numerical simulations and it gives satisfactory agreement. This analytical result helps us for the better understanding of the model. Using this result, the behavior of variable yield coefficient on outcomes of predator prey interaction in chemostat can be studied.

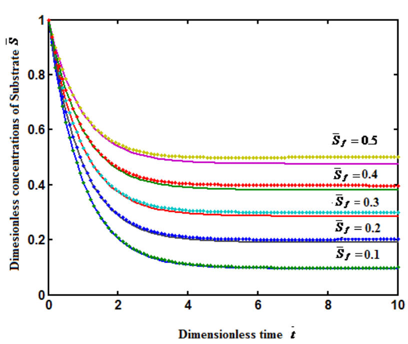

Fig. 1 Dimensionless concentration of substrate $\bar{S}$, versus dimensionless time $\bar{t}$ for various values of substrate feed concentration, $\bar{S}_f$ and for some fixed values of parameters $\bar{D}=1,\ \lambda_1=0.7$ and $\lambda=0.08$

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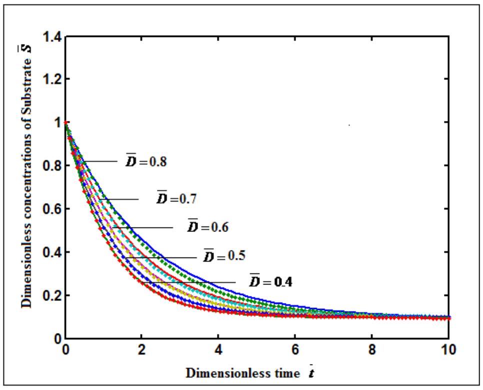


Fig. 2 Dimensionless concentration of substrate $\bar{S}$, versus dimensionless time $\bar{t}$ for various values of dilution rate, $\bar{D}$ and for some fixed values of parameters $\bar{S}_f = 0.1, \lambda_1 = 0.7$ and $\lambda = 0.08$

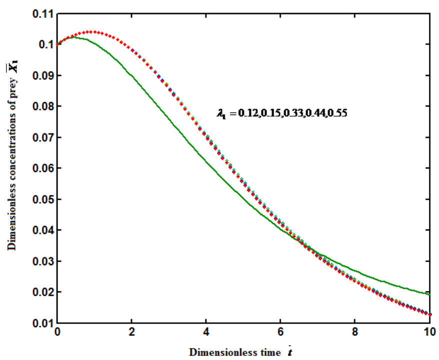


Fig. 3 Dimensionless concentration of prey $\bar{X}_1$, versus dimensionless time $\bar{t}$ for various values of kinetic parameters, λ_1 and for some fixed values of parameters $\bar{S}_f = 0.1, \bar{D} = 0.4, \bar{k} = 0.5,$ and $\lambda = 0.08.$

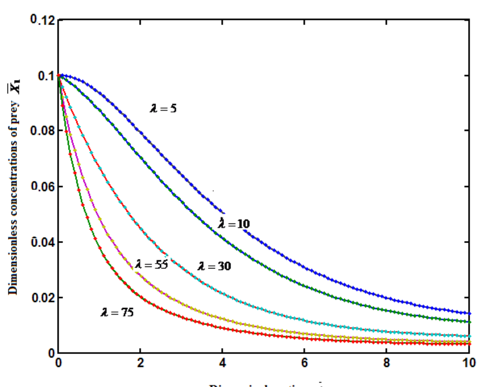


Fig. 4 Dimensionless concentration of prey $\bar{X}_1$, versus dimensionless time $\bar{t}$ for various values of kinetic parameters, λ and for some fixed values of parameters $\bar{S}_f = 0.1, \bar{D} = 0.4, \bar{k} = 0.5,$ and $\lambda_1 = 0.7$

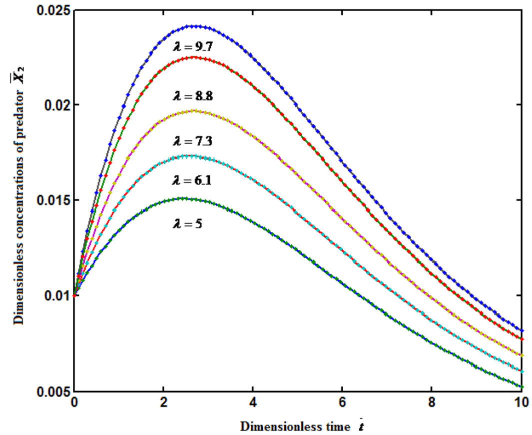
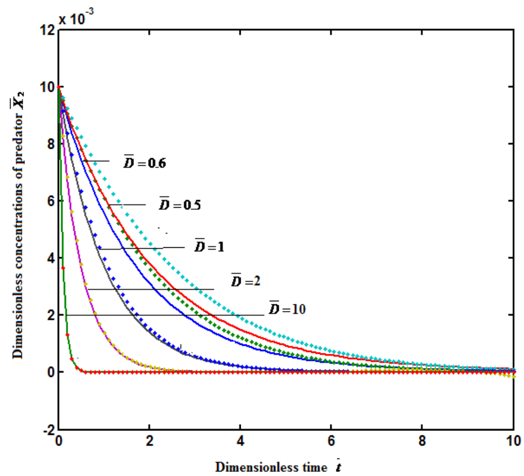


Fig. 5 Dimensionless concentration of predator $\bar{X}_2$, versus dimensionless time $\bar{t}$ for various values of kinetic parameters, λ and for some fixed values of parameters $\bar{S}_f = 0.1, \bar{D} = 0.4, \bar{k} = 0.5,$ and $\lambda_1 = 0.7$



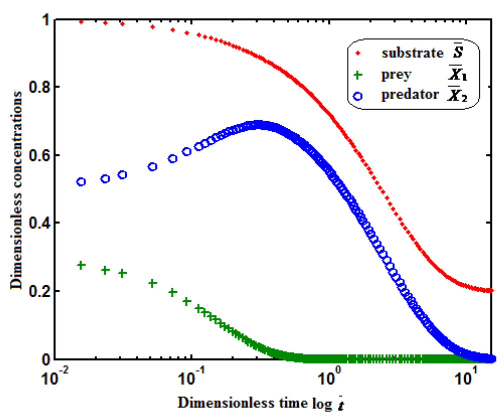


Fig. 7 Dimensionless concentration of substrate $\bar{S}$, prey $\bar{X}_1$, predator $\bar{X}_2$ versus dimensionless time $\bar{t}$ is plotted for $\bar{S}_f = 0.2, \lambda = 5, \bar{k} = 0.5, \bar{D} = 0.4$ and $\lambda_1 = 0.7$

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Appendix A

Approximate analytical solution of the nonlinear Eqns. (6) - (8) using new approach to Homotopy perturbation method

In order to solve Eqns. (6)-(8), we first construct the Homotopy as follows:

$$(1-p)\left[\frac{d\bar S}{d\bar t}+\bar D(\bar S_0-\bar S_f)+\frac{\bar S_0(1+\lambda_1)\bar X_{10}}{(1+\bar S_0)(1+\lambda_1\bar S_0)}\right] +p\left[\begin{array}{l}(1+\bar S_0)(1+\lambda_1\bar S_0)\left(\frac{d\bar S}{d\bar t}+\bar D(\bar S_0-\bar S_f)\right)\\ +\bar S_0(1+\lambda_1)\bar X_{10}\end{array}\right]=0 \qquad (A.1)$$

$$(1-p)\left[\frac{d\bar X_1}{d\bar t}+\bar D\bar X_{10}-\frac{\bar S_{zeroth}\bar X_{10}}{1+\bar S_0}+\frac{\lambda\bar X_{10}\bar X_{20}}{\bar k+\bar X_{10}}\right]+ p\left[\begin{array}{l}(1+\bar S)(\bar k+\bar X_1)\left(\frac{d\bar X_1}{d\bar t}+\bar D\bar X_1\right)\\ -\bar S\bar X_1(\bar k+\bar X_1)+\lambda\bar X_1\bar X_2(1+\bar S)\end{array}\right]=0 \qquad (A.2)$$

$$(1-p)\left[\frac{d\bar X_2}{d\bar t}+\bar D\bar X_{20}-\frac{\lambda\bar X_{1zeroth}\bar X_{20}}{\bar k+\bar X_{10}}\right] +p\left[\bar k+\bar X_1\left(\frac{d\bar X_2}{d\bar t}+\bar D\bar X_2\right)-\lambda\bar X_1\bar X_2\right]=0 \qquad (A.3)$$

The approximate solution of Eqn. (15-17) is

$$\left.\begin{array}{l}\bar S=\bar S_{zeroth}+\bar S_{first}+p^2\bar S_{second}+\ldots\\ \bar X_1=\bar X_{1zeroth}+p\bar X_{1first}+p^2\bar X_{1second}+\ldots\\ \bar X_2=\bar X_{2zeroth}+p\bar X_{2first}+p^2\bar X_{2second}+\ldots\end{array}\right\} \qquad (A.4)$$

Substituting Eqn. (A.4) into (A.1), (A.2) to (A.3) and arranging coefficients of powers p, we get

$$p^0:\frac{d\bar S_{zeroth}}{d\bar t}+\bar D(\bar S_{zeroth}-\bar S_f)+\frac{\bar S_{zeroth}(1+\lambda_1)\bar X_1}{(1+\bar S(0))(1+\lambda_1\bar S(0))}=0 \qquad (A.5)$$

$$p^0:\frac{d\bar X_{1zeroth}}{d\bar t}+\bar D\bar X_{zeroth1}-\frac{\bar S_{zeroth}\bar X_{zeroth1}}{1+\bar S(0)}+\frac{\lambda\bar X_{1zeroth}\bar X_2}{\bar k+\bar X_1(0)}=0 \qquad (A.6)$$

$$p^0:\frac{d\bar X_{2zeroth}}{d\bar t}+\bar D\bar X_{2zeroth}-\frac{\lambda\bar X_{zeroth1}\bar X_{2zeroth}}{\bar k+\bar X_1(0)}=0 \qquad (A.7)$$

The initial condition in Eqn. (10) become

$$\text{At}\quad \begin{array}{l}\bar t=0,\ \bar S_{zeroth}(0)=\bar S_0,\ \bar X_{zeroth}(0)=\bar X_{10}\\ \bar X_{2zeroth}(0)=\bar X_{20}\end{array} \qquad (A.8)$$

$$\text{And}\quad \bar S_i(0)=\bar S_0,\ \bar X_{1i}(0)=\bar X_{10},\ \bar X_{2i}(0)=\bar X_{20}\ \text{ for all } i=\textit{first, second, third}\ldots \qquad (A.9)$$

Solving Eqns. (A.5) to (A.7) for the initial condition Eqns. (A.8), we can find the following results

$$\bar S_{zeroth}(\bar t)=\frac{\bar D\bar S_f}{A}+\left(\bar S-\frac{\bar D\bar S_f}{A}\right)e^{-A\bar t} \qquad (A.10)$$

$$\bar X_{1zeroth}(\bar t)=\frac{\bar X_{10}}{1+\bar S_0}\left(\frac{a}{A_1}+\frac{be^{-A\bar t}}{A_1-A}\right) +e^{-A_1\bar t}\left(\bar X_{10}-\frac{\bar X_{10}A_2}{1+\bar S_0}\right) \qquad (A.11)$$

$$\bar X_{2zeroth}(\bar t)=\frac{\lambda\bar X_{20}}{K+\bar X_{10}}\left[\frac{\bar X_{10}\left(\frac{a}{\bar D A_1}+\frac{be^{-A\bar t}}{(A_1-A)(\bar D-A)}\right)}{1+\bar S_0}+\frac{e^{-A_1\bar t}\left(\bar X_{10}-\frac{\bar X_{10}A_2}{1+\bar S_0}\right)}{\bar D-A_1}\right] +\frac{\lambda\bar X_{20}}{K+\bar X_{10}}\left[\frac{\bar X_{10}\left(\frac{a}{\bar D-A_1}+\frac{b}{(A_1-A)(\bar D-A)}\right)}{1+\bar S_0}+\frac{\left(\bar X_{10}-\frac{\bar X_{10}A_2}{1+\bar S_0}\right)}{\bar D-A_1}\right] \qquad (A.12)$$

Appendix B: Matlab program to find the numerical solution of Eqns. (6)-(8)

```
function graphmain3
options= odeset('RelTol',1e-6,'Stats','on');
%initial conditions
Xo = [1;0.1; 0.01;];
tspan = [0 10];
tic
[t,X] =ode45(@TestFunction,tspan,Xo,options);
toc
figure
hold on
%plot(t,X(:,1),'-')
%plot(t, X(:,2),'-')
plot(t, X(:,3),'-')
legend('x1','x2','x3')
ylabel('x')
xlabel('t')
return
function [dx_dt]= TestFunction(t, x)
sf=0.1;D=10;l1=0.7;l=0.08;k=0.5;
dx_dt(1)=D*(sf-x(1))((x(1)/(1+x(1)))/((1+l1*x(1))/(1+l1)))*x(2);
dx_dt(2)=-(D*x(2))+(x(1)/(1+x(1)))*x(2)(l*x(2)/k+x(2))*x(3)
dx_dt(3)=-(D*x(3))+(l*x(2)/k+x(2))*x(3);
dx_dt = dx_dt';
return
```

NOMENCLATURE

Parameters	Description
a	Constant
b	Constant
$\bar D$	Controllable operating parameters
D	Dilution rate
K_s	Saturation constants associated with the growth values of prey
K_X	Saturation constants associated with the growth values of Predator

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$\overline{K}$	Kinetic parameters
S	Concentrations of substrate
$\overline{S}_f$	Dimensionless concentrations of substrate
S_f	Substrate feed concentration
X_1	Concentrations of prey
X_2	Concentrations of predator

Parameters	Description
$\overline{S}$	Dimensionless concentrations of substrate
$\overline{X}_1$	Dimensionless concentrations of prey
$\overline{X}_2$	Dimensionless concentrations of predator
$Y_S(S)$	Yield coefficient for prey growth on substrate
Y_P	Yield coefficient for predator growth on prey
μ_S	Maximum specific growth rates of prey
μ_P	Maximum specific growth rates of predator
λ	Kinetic parameters
λ_1	Kinetic parameters

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