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Unsteady Generalized Couette Flow of a Couple Stress Fluid between two Parallel Plates

KEYWORDS: Couple stress fluid,Couette flow, unsteady flow, Laplace transform,Numerical inversion, particle velocity.

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ABSTRACT *The flow between two parallel plates, one of which is at rest while the other is moving in its own plane with constant speed is called a simple Couette flow. The flow between two parallel plates produced by a constant pressure gradient in the direction of the flow is called a two dimensional Poiseuille flow. The generalized Couette flow is a superimposition of the simple couette flow over the two dimensional Poiseuille flow [2]. This type of flow is very important and there are many practical applications as observed by Erdogen [1]. This chapter is intended to study this problem taking couple stress fluid between the two parallel plates.*

Basic equations of an incompressible couple stress fluid flow :

The basic equations describing an incompressible couple stress fluid flow in the absence of body forces are given by [3]

$$div(\bar{q}) = 0 \tag{1}$$

$$\rho\left(\frac{\partial \bar{q}}{\partial t} + (q.\nabla \bar{q})\right) = -grad(p) - \mu\, curl(curl(\bar{q})) + \eta_1\, curl(curl(curl(curl(\bar{q})))) \tag{2}$$

where $\bar{q}$ is the velocity vector , p is the fluid pressure , μ is the viscosity coefficient and η_1 is the gyro viscosity coefficient . Let an incompressible couple stress fluid fill the region between two parallel plates at y = -h and y = h and be initially at rest . Let us consider the motion of the fluid which occurs due to the imposition of a constant pressure gradient and the simultaneous motion of the upper plate with a constant velocity U. The flow generated is assumed to be in the form

$$\bar{q} = (u(y,t), 0, 0) \tag{3}$$

The velocity vector satisfies the continuity equation and is governed by the equation

$$\rho\frac{\partial u}{\partial t} = -\left(\frac{\partial p}{\partial x}\right) + \mu\frac{\partial^2 u}{\partial y^2} - \eta_1\frac{\partial^4 u}{\partial y^4} \tag{4}$$

with the conditions

$$u(y,0) = 0 \quad -b \le y \le b \tag{.5}$$

$$\begin{aligned} u(-b,t) &= 0 \quad for\ \ all\ \ t \\ u(b,t) &= U \quad for\ \ t > 0^+ \end{aligned} \tag{.6}$$

and

$$\begin{aligned} \frac{\partial p}{\partial x} &= 0 \quad for\ \ t < 0 \\ &= -G^* \quad for\ \ t > 0^+ \end{aligned} \tag{7}$$

Introducing the non-dimenssionalization scheme given by

$$y = b\tilde{y} \quad x = b\tilde{x} \quad u = U\tilde{u} \quad t = \frac{b}{U}\tilde{t} \quad P = \frac{\mu U}{b}\tilde{P} \tag{8}$$

and defining

$$R = \frac{\rho U b}{\mu} \qquad \frac{\eta}{b^2 \mu} = a^2 \tag{9}$$

after dropping the tildes the equation (4) reduces to,

$$R\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} - a^2\frac{\partial^4 u}{\partial y^4} \tag{10}$$

Then we have to solve the equation

$$R\frac{\partial u}{\partial t} = G^* + \frac{\partial^2 u}{\partial y^2} - a^2\frac{\partial^4 u}{\partial y^4} \tag{11}$$

subject to the conditions (a) :

$$\begin{aligned} &u(1,t) = 1 \quad u(-1,t) = 0 \quad for \quad t > 0^+ \\ &u(y,0) = 0 \quad for \quad -1 \le y \le 1 \quad and\ \ t \le 0 \end{aligned} \tag{12}$$

$$and \quad \frac{\partial u}{\partial y} = 0 \quad for \quad y = \pm 1 \quad and\ t > 0 \tag{13}$$

and subject to the conditions (b) :

$$\begin{aligned} &u(1,t) = 1 \quad u(-1,t) = 0 \quad for \quad t > 0^+ \\ &u(y,0) = 0 \quad for \quad -1 \le y \le 1 \quad and\ \ t \le 0 \end{aligned} \tag{14}$$

$$and \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad for \quad y = \pm 1 \quad and\ t > 0^+ \tag{15}$$

As $t \to \infty$, it is natural that the flow becomes steady. Hence the equation (11) becomes

$$a^2\frac{d^4 u}{dy^4} - \frac{d^2 u}{dy^2} - G^* = 0 \tag{16}$$

since $u = u(y)$.

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To get the steady flow field we have to solve the above equation and implement the boundary conditions (a) or conditions (b) as explained in the introduction.

The equation (16) can be rewritten in the form

$$D^2\,(D^2 - \frac{1}{a^2})u \;=\; \frac{G^*}{a^2} \tag{17}$$

where

$$u = u(y) \quad \text{and} \quad D = \frac{d}{dy}$$

It is seen that the most general solution of equation (17) is given by

$$u \;=\; A + B\,y + Ce^{y/a} + D\,e^{-y/a} - G^*\,\frac{y^2}{2} \tag{18}$$

Implementing the conditions (a), after a series of routine calculations, the steady state velocity is seen to be

$$u_{st}\,(y) = \frac{1}{2} + (1 - y^2 - 2a\coth 1/a)\frac{\cosh 1/a}{2(\cosh 1/a - a\sinh h\; 1/a}\; y + \frac{a\; G^*\,(a\sinh\; 1/a - \cosh\; 1/a)(\cosh\; y/a) + \frac{a}{2}\sinh\; 1/a \sinh\; y/a}{\sinh\; 1/a\,(a\sinh\; 1/a - a\cosh\; 1/a)} \tag{19}$$

Using conditions (b) the steady state velocity u is obtained as

$$u_{st}\,(y) = \frac{1 + y + G^*\,((1 - y^2 - 2a^2)}{2} + \frac{a^2\,G^*\;\cosh\; y/a}{\cosh 1/a}\; y \tag{20}$$

General problem using conditions (a) :

As we have to solve the general unsteady problem governed by equations (11), (12) and (13) we assume that

$$u(y,t) \;=\; f\,(y,t) + u_{st}\,(y) \tag{21}$$

hence $u_{st}(y)$ is given by equation(19) with $u(y,t) \to u(y)\;\; as\;\; t \to \infty$.

This implies that $f\,(y,t) \to 0\;\; as\;\; t \to \infty$.

Further the condition u(-1,t) = 0 yields

$$\text{f(-1,t)} = 0 \tag{22}$$

The condition u(1,t) = 1 implies that

$f\,(1,t) + u_{st}\,(1) \;=\; 1$ and as $u_{st}\,(1) = 1$, we get

$$f\,(1,t) \;=\; 0 \tag{23}$$

Using equation (21) in equation (10), since conditions (a) needs $\frac{\partial u}{\partial y} = 0\;\; on\;\; y = \pm 1$, we

get $\frac{\partial f}{\partial y} = 0\;\; on\;\; y = \pm 1$

We notice that f(y,t) is governed by the differential equation

$$R\,\frac{\partial f}{\partial t} \;=\; \frac{\partial^2 f}{\partial y^2} - a^2\,\frac{\partial^4 f}{\partial y^4} \tag{24}$$

subject to the boundary conditions (a)

$$f\,(\pm 1,t) \;=\; 0 \qquad \left(\frac{\partial f}{\partial y}\right)_{y=\pm 1} = 0 \tag{25}$$

The equation (21) implies that

$$f\,(y,0) \;=\; u(y,0) - u_{st}\,(y) \tag{26}$$

As u(y,0) = 0, this results in

$$f\,(y,0) \;=\; -u_{st}\,(y) \tag{27}$$

Hence we have to solve equation (24) subject to the initial condition in equation (27) ar boundary conditions in equation (25).

Taking Laplace transform of equation (24) and using equation (27), we g

$$\frac{\partial^4 \bar{f}}{\partial y^4} - \frac{1}{a^2}\frac{\partial^2 \bar{f}}{\partial y^2} + \frac{Rs}{a^2}\,\bar{f} = -\frac{Ru_{st}}{a^2} \tag{28}$$

This can be rewritten as

$$(D^2 - \alpha^2)(D^2 - \beta^2)\,\bar{f} \;=\; -\frac{R\,u_{st}}{a^2} \tag{29}$$

where u_{st} for the present case is given by equation (19) where

$$\alpha^2 + \beta^2 = \frac{1}{a^2} \quad ; \quad \alpha^2\beta^2 = \frac{Rs}{a^2} \tag{30}$$

The solution of equation (29) by elementary but straight forward calculation is given by

$$u \;=\; A\,e^{\alpha y} + B\,e^{-\alpha y} + Ce^{\beta y} + D\,e^{-\beta y} + P(y,s) \tag{31}$$

where

$$P(y,s) = -\frac{1}{2s}\left\{1 + G^*(1 - 2a\coth {}^{1}\!/_{a} - y^2 - \frac{2}{Rs})\right\} + \frac{1}{2s}\,\frac{\cosh {}^{1}\!/_{a}}{a\sinh {}^{1}\!/_{a} - \cosh {}^{1}\!/_{a}}\,y - \frac{2a}{4s}\left\{\frac{\sinh {}^{1}\!/_{a}\sinh {}^{y}\!/_{a} + 2G^*(a\;\sinh {}^{1}\!/_{a} - \cosh {}^{1}\!/_{a})\cosh {}^{y}\!/_{a}}{\sinh {}^{1}\!/_{a}(a\;\sinh {}^{1}\!/_{a} - \cosh {}^{1}\!/_{a})}\right\} \tag{32}$$

In equation (31) *A,B,C,D* are arbitrary constants which are to be determined subject to the conditions obtained by taking the Laplace Transforms of those in equation (25)

Thus *A, B, C, D* are to be obtained from equations (31), (32) implementing the condition:

$$\bar{f}\,(\pm 1, s) \;=\; 0 \qquad \frac{\partial \bar{f}(\pm 1, s)}{\partial y} = 0 \tag{33}$$

Once again, using equation (31) and equation(33), we notice that the constants *A,B,C,D* can be determined, by solving the simultaneous linear equations in *A,B,C,D* given by

$$A\,e^{\alpha} + B\,e^{-\alpha} + Ce^{\beta} + D\,e^{-\beta} = -\left(\frac{G^*}{Rs^2} - \frac{1}{s}\right) \tag{34}$$

$$A\,e^{-\alpha} + B\,e^{\alpha} + Ce^{-\beta} + D\,e^{\beta} = -\left(\frac{G^*}{Rs^2}\right) \tag{35}$$

$$A\alpha\,e^{\alpha} - B\alpha\,e^{-\alpha} + C\beta\,e^{\beta} - D\,\beta\,e^{-\beta} = 0 \tag{36}$$

$$A\alpha\,e^{-\alpha} - B\alpha\,e^{\alpha} + C\beta e^{-\beta} - D\,\beta e^{\beta} = 0 \tag{37}$$

Solving the equations (34) – (37) for the unknowns *A,B,C,D* we get

$$A = \frac{-Q_1\,\beta\sinh\beta}{2\,R_1} + \frac{Q_2\,\beta\cosh\beta}{2\,R_2} \qquad B = \frac{-Q_1\,\beta\sinh\beta}{2\,R_1} - \frac{Q_2\,\beta\cosh\beta}{2\,R_2} \qquad C = \frac{Q_1\,\alpha\;\sinh\alpha}{2\,R_1} - \frac{Q_2\,\alpha\cosh\alpha}{2\,R_2} \qquad D = \frac{Q_1\,\alpha\;\sinh\alpha}{2\,R_1} + \frac{Q_2\,\alpha\cosh\alpha}{2\,R_2} \tag{38}$$

where

$$Q_1 = \frac{G^*}{R\,s^2} - \frac{1}{2\,s} \qquad Q_2 = \frac{1}{2\,s}$$

$$R_1 = \beta\cosh\alpha\sinh\beta - \alpha\sinh\alpha\cosh\beta$$

$$R_2 = \beta\;\;\sinh\alpha\cosh\beta - \alpha\cosh\alpha\sinh\beta$$

And thus $\bar{f}(y,s)$ and hence $\bar{u}(y,s)$ are completely determined. The expression for $\bar{u}\,(y,s)$ can be seen to be

$$\bar{u}(y,s) = \left(\frac{1}{2s} - \frac{G^*}{Rs^2}\right)\left(\frac{\beta\sinh\beta\cosh\alpha y - \alpha\sinh\alpha\cosh\beta y}{\beta\sinh\beta\cosh h\,\alpha - \alpha\sinh\alpha\cosh\beta}\right) + \frac{1}{2s}\left(\frac{\beta\cosh\beta\sinh\alpha y - \alpha\cosh\alpha\sinh\beta y}{\beta\cosh\beta\sinh\alpha - \alpha\cosh\alpha\sinh\beta}\right) + \frac{G^*}{Rs^2} \tag{39}$$

General problem using conditions (b) :

The conditions (b) proposed by V.K.Stokes as mentioned earlier, are

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$$u(\pm 1, t) = 0 \tag{40}$$

and

$$\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } y = \pm 1 \text{ and for all } t > 0^+ \tag{41}$$

Hence, we have to solve the general unsteady problem governed by equations (11) and subject to the boundary conditions given in equations (13) and (14).

Here again, we assume

$$u(y,t) = f(y,t) + u_{st}(y) \tag{42}$$

where $u_{st}(y)$ is given by equation (20) with the assumption that

$$u(y,t) \to u_{st}(y) \qquad \text{as } t \to \infty .$$

Further the conditions in equation (13) and (14) as proceeding in case (a) yield

$$f(\pm 1, t) = 0 \text{ , } f(y,0) = -u_{st}(y) \tag{43}$$

and

$$\frac{\partial^2 f(\pm 1, t)}{\partial y^2} = 0 \tag{44}$$

Taking Laplace transform of the equation (11) , using equations (42), (43), (44) and the initial conditions

$$\bar{f}(\pm 1, t) = 0 \text{ , } \quad \frac{\partial^2 \bar{f}(\pm 1, t)}{\partial y^2} = 0 \tag{45}$$

we get

$$\bar{f}(y,s) = A e^{\alpha y} + B e^{-\alpha y} + C e^{\beta y} + D e^{-\beta y} + P(y,s)$$

where here

$$P(y,s) = -\frac{1}{2s}\left(1 + y + G^*(1 - 2a^2 - y^2 - \frac{2}{Rs}\right) - \frac{G^* a^2}{s}\frac{\cosh y/a}{\cosh 1/a}$$

Implementing the conditions in equation (45), we get

$$A = \frac{[-Q_1 \sinh\alpha + Q_2 \cosh\alpha]\beta^2}{2(\beta^2 - \alpha^2)\sinh\alpha\cosh\alpha}$$

$$B = \frac{[-Q_1 \sinh\alpha - Q_2 \cosh\alpha]\beta^2}{2(\beta^2 - \alpha^2)\sinh\alpha\cosh\alpha}$$

$$C = \frac{[Q_1 \sinh\beta - Q_2 \cosh\beta]\alpha^2}{2(\beta^2 - \alpha^2)\cosh\beta\sinh h\beta} \tag{46}$$

$$D = \frac{[Q_1 \sinh\beta + Q_2 \cosh\beta]\alpha^2}{2(\beta^2 - \alpha^2)\cosh\beta\sinh h\beta}$$

Thus $\bar{f}(y,s)$ is completely determined and hence with the conditions (b), $\bar{u}(y,s)$ is given by

$$\bar{u}(y,s) = \frac{1}{(\beta^2 - \alpha^2)}\left\{\begin{array}{l}\left[\frac{1}{2s} - \frac{G^*}{Rs^2}\right]\left[\frac{\sinh\alpha\ \cosh\alpha y}{\sinh\alpha\cosh\alpha}\beta^2 - \frac{\sinh\beta\cosh\beta y}{\sinh\beta\cosh\beta}\alpha^2\right] + \\ \frac{1}{2s}\left[\frac{\cosh\alpha\ \sinh\alpha y}{\cosh\alpha\sinh\alpha}\beta^2 - \frac{\cosh\beta\sinh\beta y}{\cosh\beta\sinh\beta}\alpha^2\right]\end{array}\right\} + \frac{G^*}{Rs^2} \tag{47}$$

The skin friction on the plates is given by

$$\bar{t}_{yx} = \left(\frac{\partial\bar{u}}{\partial y} - \frac{\partial^3\bar{u}}{\partial y^3}\right)_{on\ y = \pm 1} \tag{48}$$

For condition (a) for $y = \pm 1$

$$\begin{aligned}-\frac{\partial^3\bar{u}}{\partial y^3} = &-\left(\frac{1}{2s} - \frac{G^*}{Rs^2}\right)\left(\frac{\alpha^3\beta\sinh\beta\cosh\alpha y - \beta^3\alpha\sinh\alpha\cosh\beta y}{\beta\sinh\beta\cosh h\alpha - \alpha\sinh\alpha\cosh\beta}\right) \\ &+ \frac{1}{2s}\left(\frac{\alpha^3\beta\cosh\beta\sinh\alpha y - \beta^3\alpha\cosh\alpha\sinh\beta y}{\beta\cosh\beta\sinh\alpha - \alpha\cosh\alpha\sinh\beta}\right)\end{aligned} \tag{49}$$

For condition (b) for $y = \pm 1$

$$= \frac{1}{(\beta^2 - \alpha^2)}\left\{\begin{array}{l}\left[\frac{1}{2s} - \frac{G^*}{Rs^2}\right]\left[-\frac{\sinh\alpha\ \sinh\alpha y}{\sinh\alpha\cosh\alpha}\alpha(1+\alpha^2)\beta^2 + \frac{\sinh\beta\sinh\beta y}{\sinh\beta\cosh\beta}\beta(1+\beta^2)\alpha^2\right] + \\ \frac{1}{2s}\left[\frac{\cosh\alpha\ \cosh\alpha y}{\cosh\alpha\sinh\alpha}\alpha(1+\alpha^2)\ \beta^2 - \frac{\cosh\beta\cosh\beta y}{\cosh\beta\sinh\beta}\beta(1+\beta^2)\alpha^2\right]\end{array}\right\} \tag{50}$$

Numerical results and discussion :

The expression $\bar{u}(y,s)$ is inverted numerically for each of the conditions (a) and (b) for different values of y and different values of t for diverse values of the parameters a, Gand R . The variation of u(y,t) is displayed through graphs.

In figure (1) we have presented the variation of velocity with the distance y from the stationary plate to the moving plate for diverse values of time t with G=2, a=0.5 and R=5. For any fixed y as time increases, we observe that the velocity is increasing as can be expected from the physics of the problem.

In figure (2) for a fixed t=1, pressure gradient G=2 and Reynolds number R=5, we show the variation of velocity with respect to y as the couple stress parameter increases. An increase in couple stress parameter indicates an increase in the effect of couple stresses. From figure (2) as a increases ,nearer to the stationary plate, initially as y increases there is a decreas in velocity. However as y is increased this trend is reversed. That is as "a" increases, nearer to the moving plate, for any y velocity shows an increasing trend. i.e as y increases from -1 to +1 , there is a critical value of y where the initial decreasing trend of the velocity is reversed and the velocity shows a continuous increasing trend. This is in tune with the observation made by V.K.Stokes [3] while discussing the Couette flow of a couple stress fluid between two parallel plates where the lower one is stationary and the upper one is moving with a constant velocity.

In figure (3) we have plotted the variation of velocity with different values of the pressure gradient when t=1 , a=0.5 and R=5 . As G increases, the velocity shows at any point an increasing trend.

Figure (4) shows the variation of velocity with distance as R varies while t,a and,G are fixed as 1, 0.5 and 2 respectively . Here as R increases for any y, the velocity decreases.

Figures (5) (6) (7) and (8) respectively show, the variation of velocity with distance while time varies, parameter "a" varies, pressure gradient parameter G varies and Reynolds number R varies respectively for boundary condition (b). The results in this case are qualitatively similar to those obtained for boundary conditions (a). This is similar to the observations made by Devakar [4] while considering generalized Stoke's problems for a couple stress fluid.

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boundary Condition (A)

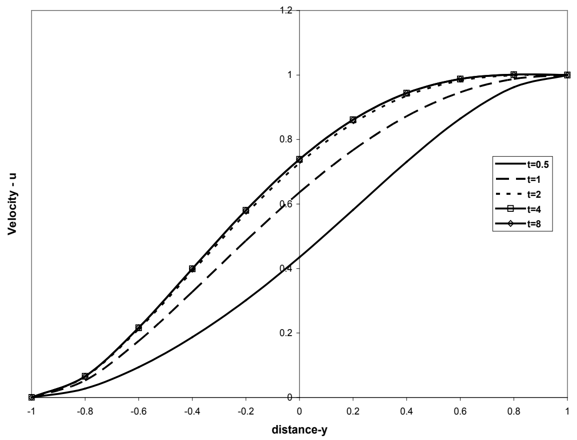

Figure (1): Variation of velocity with distance for various values of t when G = 2; a = 0.5; R = 5.

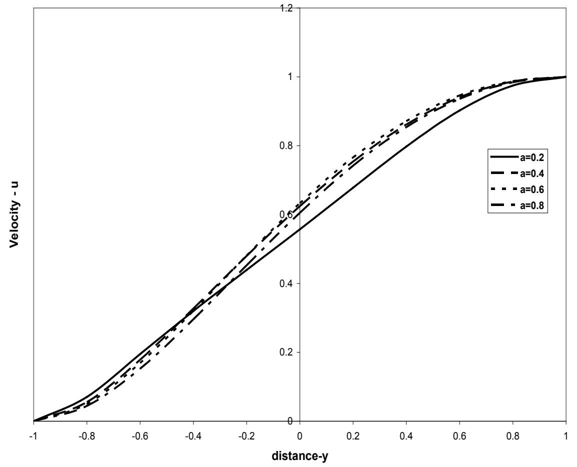

Figure (2): Variation of velocity with distance for various values of a when t = 1; G = 2; R = 5.

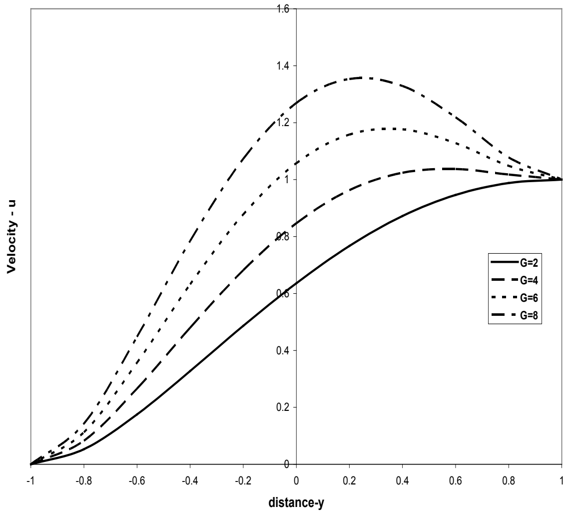

Figure (3): Variation of velocity with distance for various values of G when t = 1; a = 0.5; R = 5

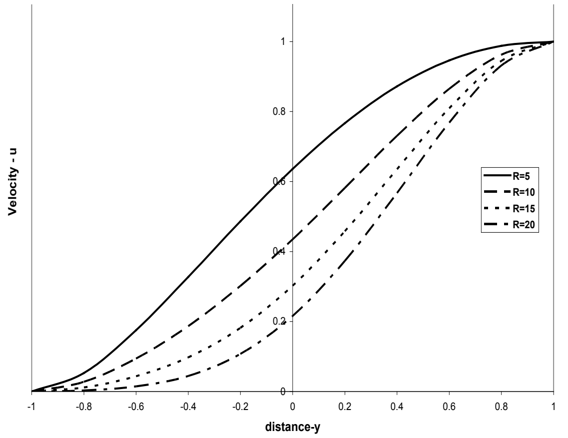

Figure (4) : Variation of velocity with distance for various values of R when t = 1; a = 0.5; G = 2

Boundary Condition (B)

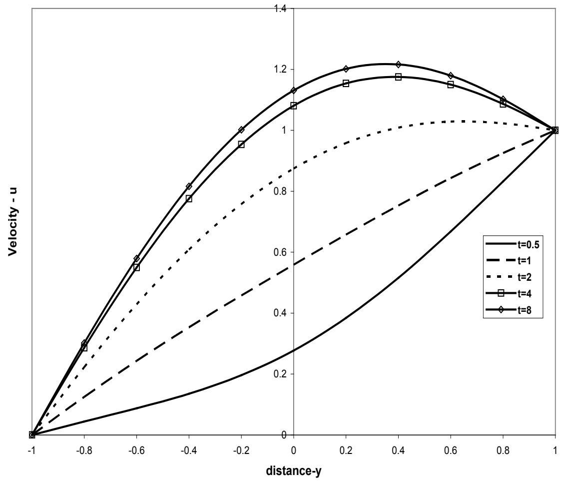

Figure (5) : Variation of velocity with distance for various values of t when *G = 2; a = 0.5; R = 5.*

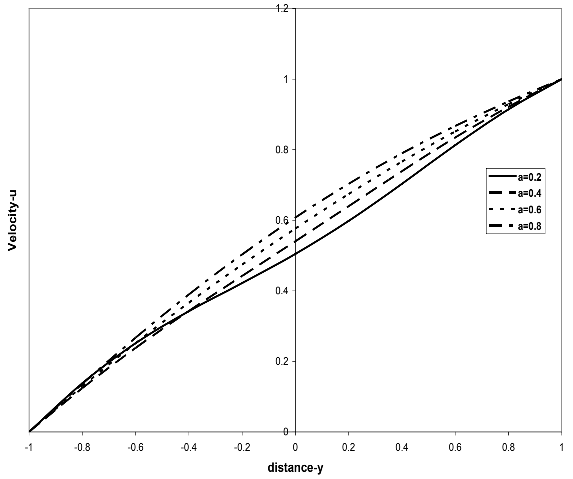

Figure (6) : Variation of velocity with distance for various values of a when *t = 1; G = 2; R = 5.*

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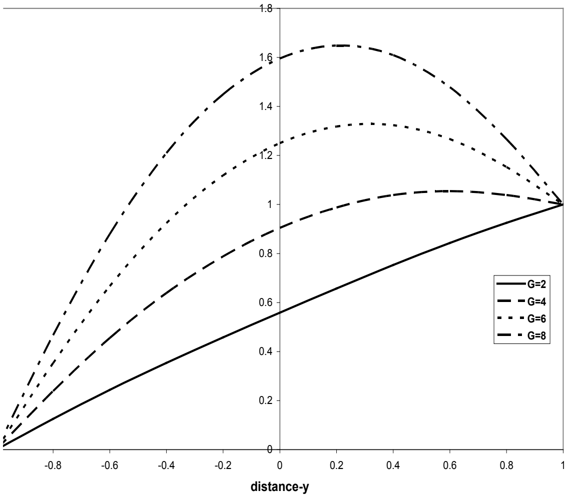

Figure (7) : Variation of velocity with distance for various values of G when *t* = 1; a = 0.5; R = 5.

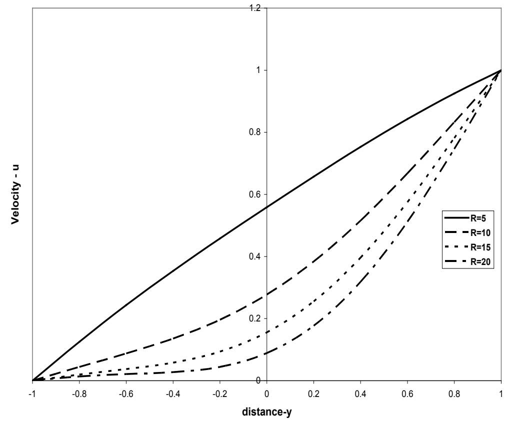

Figure (8) : Variation of velocity with distance for various values of R when *t* = 1; a = 0.5; G = 2.

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