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# μ -WEAK STRUCTURES

KEYWORDS: μ – weak structure; $\omega_\mu$ – closure operator ; $\omega_\mu$ – open set.

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**ABSTRACT** *Császár in [4] introduce weak structures as generalization of general topology, and introduce closure and interior, also introduce some types of ω -open set. In this paper, we define μ - weak structures as a generalization of weak structures and introduced some types of $\omega_\mu$ _ open sets. The relation between these types are introduced.*

## 1 Introduction

Let $X$ be a non-empty set and P its power set. A structure on $X$ is a subset of $P$ (e.g. a topology on $X$), and an operation on $X$ is a function from $P$ to $P$. If $\varphi$ is an operator on $X$, and $A\subset X$, we write $\varphi A$ instead of $\varphi(A)$, and if $\psi$ is another operator, we write $\phi\psi$ instead of $\phi$ o $\psi$. If $v$ is a structure on $X$, we call $v$ -open the elements of $v$ and $v$ -closed their complements (in $X$). If $v$ is a suitable structure in $X$ and $\subset X$ , we denote by $i_v A$ the union of all $v$ -open sets contained in $A$ and by $c_v A$ the intersection of all $v$ -closed sets containing $A$. In the literature, we find $i_v A$ and $c_v A$ in the case, of course, if $v$ is a topology, but also in the case if $v$ is a generalized topology in the sense of [2] (i.e. a $g\subset P$ such that $\emptyset\in g$ and the union of any subfamily of $g$ always belongs to ), or a minimal structure in the sense of [5] (i.e. $m\subset P$ such that $\phi\in m$ and $X\in m$ ). The purpose of this paper is to define; the family $\omega_\mu=\{v\cap\mu : v\,\epsilon\,\omega\}$ is the $\omega_\mu$-structure induced over $\mu\subset X$ by $\omega$ (by short $\omega_\mu S$). The elements of $\omega_\mu$ are called $\omega_\mu$-open sets; a set $v$ is a $\omega_\mu$-closed set if $\mu-v\,\epsilon\,\omega_\mu^c$ . We note that $\omega_\mu^c$ is the family of all $\omega_\mu$-closed sets.

## 2 $i_\mu\omega$ and $c_\mu\omega$

In this section we define $i_\mu\omega$ and $c_\mu\omega$ under more general conditions and to show that the important properties of these operations remain valid under these conditions.

Let $v\in\omega_\mu S$ we define the $\omega_\mu$-interior (by short $i_\mu\omega$ ) of $\mu$ as the finest $\omega_\mu$ - open sets contained in $v$ that is, $i_\mu\omega(v)=\bigcup_{\zeta\in\omega_\mu}\{\zeta: \zeta\subseteq v\}$.

Let $v\in\omega_\mu S$ we define the $\omega_\mu$-closure (by short $c_\mu\omega$ ) of $\mu$ as the smallest $\omega_\mu$-closed sets which contained in $v$ that is, $c_\mu\omega(v)=\bigcup_{\zeta\in\omega_\mu}\{\mu-\zeta: v\subseteq\mu-\zeta\}$.

**Example 2.1.**
Let $X=\{a,b,c\}$, $\omega=\{\phi,\{a\},\{a,b\}\}$ , $\mu=\{a,b\}$, and $v=\{c\}$. Then $i_\mu\omega(v)=\phi$, and $i_\mu\omega(v)=\{c\}$.

***Remark* 2.1.**
Every $\omega_\mu$ -open set is $\omega$ -open set, but the converse is not true.

**Example 2.2.** Let $X=\{a,b,c,d,e\}$, $\omega=\{\phi,\{a\},\{b\},\{c\},\{d\}\}$ , $\mu=\{a,b,c,d\}$, and $v=\{a,b\}$. Then $v$ is $\omega_\mu$ – open but not $\omega$ – open set.

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**Proposition 2.1.**

Let $\omega_\mu$ be a weak structure of $X$, and $A, B, \mu \subseteq X$. Then the following are hold:-

i. $i_\mu\omega(A \cap B) \subset i_\mu\omega(A) \cap i_\mu\omega(B)$

ii. $c_\mu\omega(A) \cup c_\mu\omega(B) \subset \cap c_\mu\omega(A \cup B)$

iii. If $A \in \omega_\mu S$, then $i_\mu\omega(A) = A$, and $A \in \omega_\mu^c S$, then $A = c_\mu\omega(A)$.

**Proof.**

It's obvious.

**Proposition 2.2.**

Let $\omega_\mu$ be a weak structure of $X$, and $v, \mu \subseteq X$. Then $v = i_\mu\omega(v)$, and if $v$ is $\omega_\mu$ – closed then $v = c_\mu\omega(v)$.

***Remark* 2.2.**

The conversely of the above is not true, i.e., if $v = i_\mu\omega(v)$ does not imply $v$ is $\omega_\mu$- open. And $v = c_\mu\omega(v)$ does not imply that $v$ is $\omega_\mu$-closed.

**Example 2.2.**

Let $X = \{a, b, c, d, e\}$, $\omega = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}\}$, $\mu = \{a, b, c, d\}$, and $v = \{a, b\}$. Then $i_\mu\omega(v) = v = c_\mu\omega(v)$. But $v \notin \omega_\mu$ and $v$ is not $\omega_\mu$– closed set.

**Theorem 2.1.**

Let $\omega_\mu$ be a weak structure of $X$, and $v, \alpha, \mu \subseteq X$. then :

(1) $i_\mu\omega(v) \subset v \subset c_\mu\omega(v)$

(2) $v \subset \alpha$ implies $i_\mu\omega(v) \subset i_\mu\ \omega(\alpha)$, and $c_\mu\omega(v) \subseteq c_\mu\omega(\alpha)$

(3) $i_\mu\omega(v) = i_\mu\omega\big(i_\mu\omega(v)\big)$, and $c_\mu\omega\left(c_\mu\omega(v)\right) = c_\mu\omega(v)$

(4) $i_\mu\omega(\mu - v) = \mu - c_\mu\omega(v)$, and $c_\mu\omega(\mu - v) = \mu - i_\mu\omega(v)$.

**Proof.**

(1) $i_\mu\omega(v) = \bigcup\{v : v \in \omega_\mu S, v \subseteq v\} \subseteq v \subseteq \bigcap\{\mu - v : v \in \omega_\mu S, v \subseteq \mu - v\} = c_\mu\omega(v)$

(2) If $\omega$ is $\omega_\mu$- open , $T$ is $\omega_\mu$- closed. And $\omega \subseteq v(\alpha \subset T)$, then $\omega \subseteq \alpha(v \subset T)$, thus [2] is true.

(3) If $\omega$ is $\omega_\mu$- open, then $\omega \subseteq v$ iff $\omega \subset i_\mu\omega(v)$ (from (1), (2), and the definition of $i_\mu\omega(A)$) . This implies the first part of (3). **Similarly:** a $\omega_\mu$- closed $T$ contains $c_\mu\omega(v)$ *iff* $T$ contains $v$, hence the second part is also true.

(4) Finally, results from the fact that $W$ is $\omega_\mu$- open *iff* $\omega_\mu$ is $\mu_\omega$- closed. □

***Remark* 2.3.**

Both $i_\mu\omega$ and $c_\mu\omega$ are monotonic by (1) from above theorem. And by (3) in the same theorem both are idempotent in the sense of [1]. Now consider $A \subset X$ and $x \subset X$.

**Lemma 2.1.** Let $\omega_\mu$ be a weak structure of $X$, $x \in i_\mu\omega(A)$, iff there is a $\omega_\mu$ - open set $W \subseteq A$ such that $x \in W$.

**Remark 2.4.**

A somewhat more complicated argument describes conditions for $x \in c_\mu\omega(A)$.

**Lemma 2.2.**

Let $\omega_\mu$ be a weak structure of $X$, we have $x \in c_\mu\omega(A)$ iff $W \cap A \neq \emptyset$ where ever $x \in W \in \omega_\mu$.

**Proof.**

Clearly $x \notin c_\mu\omega(A)$ iff there is a $\omega_\mu$ -closed set $T$ such that $A \subset T$ and $x \notin T$. Put

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$W = \mu - T$, then $W$ is $\omega_\mu$ - open, $x \in W$ and $W \cap A = \emptyset$. Thus $x \notin c_\mu\omega(A)\ iff$ there is a $W \in \omega_\mu$ such that $x \in W$ and $W \cap A = \emptyset$. **Finally**:

$x \in c_\mu\omega(A)\ iff$ among the $\omega_\mu$- open sets $W$ containing $x$, there is no one for which $W \cap A = \emptyset$, i.e. $W \cap A \neq \emptyset$ _ for all these $W$.

Let $\omega_\mu$ be a weak structure of $X$, and $\mu \subseteq X$, and consider the functions from $P$ to $P$ having the form $\varphi_1 \circ \varphi_2 \circ \ldots \varphi_n$ where $\varphi_k = i_\mu\omega$ or $\varphi_k = c_\mu\omega$ for all $k$.
**Observe**:

e.g. $i_\mu\omega(i_\mu\omega) = i_\mu\omega$ and $c_\mu\omega(c_\mu\omega) = c_\mu\omega$ see theorem (2.1.3).

**Proposition 2.3.**

Let $\omega_\mu$ be a weak structure of $X$ and $\mu \subseteq X$. Then :
$c_\mu\omega(i_\mu\omega(c_\mu\omega(i_\mu\omega))) = c_\mu\omega(i_\mu\omega)$, and $i_\mu\omega(c_\mu\omega(i_\mu\omega(c_\mu\omega))) = i_\mu\omega(c_\mu\omega)$.

# 3 The structures of $\omega_\mu - so, \omega_\mu - sp, \omega_\mu - ssp, \omega_\mu$-$\gamma O$, $\omega_\mu - po$

**Definition 3.1.**

Let $\omega_\mu$ be a weak structure of $X$ and $\mu \subseteq X$. Then $v$ is called:

(1) An $\omega_\mu$ -semi open (briefly, $\omega_\mu - so$) set if there exists $\lambda \in \omega_\mu$ such that $\lambda \subset c_\mu\omega(\lambda)$
(or,$v \subseteq c_\mu\omega(i_\mu\omega)$;

(2) An $\omega_\mu$ -preopen (briefly, $\omega_\mu - po$) set if $v \subseteq i_\mu\omega(c_\mu\omega(v))$;

(3) An $\omega_\mu$ -strongly semi open (briefly, $\omega_\mu - sso$) set if there exists $\lambda \in \omega_\mu$ such that
$\lambda \subseteq v \subseteq i_\mu\omega(c_\mu\omega(\lambda))$
(or,$v \subseteq i_\mu\omega(c_\mu\omega(i_\mu\omega(v)))$);

(4) An $\omega_\mu$ - semi preopen (briefly, $\omega_\mu - spo$) set if there exists an $\omega_\mu - po$ set $\lambda$ such that
$\lambda \subseteq v \subseteq c_\mu\omega(\lambda)$
(or,$v \subseteq c_\mu\omega(i_\mu\omega(c_\mu\omega(v)))$ );

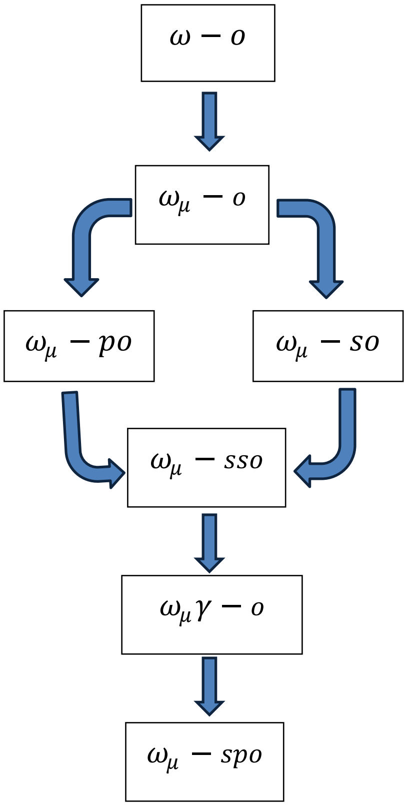
(5) For a set $\subseteq X$ ; let $A \in \omega_\mu\gamma -$ open set if $A \subseteq i_\mu\omega(c_\mu\omega(A)) \cup c_\mu\omega(i_\mu\omega(A))$.

Their complements are called $\omega_\mu$ −semiclosed (briefly, $\omega_\mu - sc$), $\omega_\mu$ −preclosed (briefly, $\omega_\mu - pc$), $\omega_\mu$ −strongly semi closed (briefly, $\omega_\mu - ssc$), $\omega_\mu$ −semin pre closed (briefly, $\omega_\mu - spc$) set.

$\omega_\mu - so$, $\omega_\mu - po$, $\omega_\mu - sso$, and $\omega_\mu - spo$ (resp. $\omega_\mu - sc$, $\omega_\mu - pc$, $\omega_\mu - rc$, $\omega_\mu - ssc$ and $\omega_\mu - spc$) will always denote the family of $\omega_\mu$ −semiopen, $\omega_\mu$ −preopen, $\omega_\mu$ −strongly semiopen and $\omega_\mu$ −semipreopen (resp. $\omega_\mu$ −semiclosed, $\omega_\mu$ −preclosed, $\omega_\mu$ −regular closed, $\omega_\mu$ −strongly semiclosed and $\omega_\mu$ −semipreclosed) sets.

**Proposition 3.1.**

The implications between these different notions of $\omega_\mu$ sets are given by the following diagram:




**Proof.**

Let $\omega_\mu$ be a weak structure of $X$, consider a set $v \in \omega_\mu$. Then $v = i_\mu\omega(v)$ by **Proposition (2.2)** then $i_\mu\omega(v) \subset c_\mu\omega\left(i_\mu\omega(v)\right)$ by **Theorem (2.1.1)**, hence $v \subset c_\mu\omega\left(i_\mu\omega(v)\right)$. As $v$ is $\omega_\mu$-open, this implies $v \subset i_\mu\omega\left(c_\mu\omega\left(i_\mu\omega(v)\right)\right)$. Thus $v \in \omega_\mu - sso$.

**Similarly**:

$i_\mu\omega\left(c_\mu\omega\left(i_\mu\omega(v)\right)\right) \subset c_\mu\omega\left(i_\mu\omega(v)\right)$.

by **Theorem (2.1.1)**. So that $v \subset i_\mu\omega\left(c_\mu\omega\left(i_\mu\omega(v)\right)\right)$ implies $v \subset c_\mu\omega\left(i_\mu\omega(v)\right)$ so $\omega_\mu - sso \subset \omega_\mu - po$.

**Clearly** $v \subset c_\mu\omega\left(i_\mu\omega(v)\right)$ implies

$v \subset \left(i_\mu\omega\left(c_\mu\omega(v)\right)\right) \cup c_\mu\omega\left(i_\mu\omega(v)\right)$

so that

$i_\mu\omega\left(c_\mu\omega(v)\right) \cup c_\mu\omega\left(i_\mu\omega(v)\right) \subset \omega_\mu - \gamma o$.

As

$c_\mu\omega\left(i_\mu\omega(v)\right) \subset c_\mu\omega\left(i_\mu\omega\left(c_\mu\omega(v)\right)\right)$

by **theorem (2.1.1)** and **(2.1.2),** moreover

$i_\mu\omega\left(c_\mu\omega(v)\right) \subset c_\mu\omega\left(i_\mu\omega\left(c_\mu\omega(v)\right)\right)$

by **theorem (2.1.1).** We have

$$i_\mu\omega\left(c_\mu\omega(v)\right) \cup c_\mu\omega\left(i_\mu\omega(v)\right) \subset c_\mu\omega\left(i_\mu\omega\left(c_\mu\omega(v)\right)\right).$$

and $\omega_\mu - \gamma o \subset \omega_\mu - spo$.

**Similarly**

$i_\mu\omega\left(c_\mu\omega\left(i_\mu\omega(v)\right)\right) \subset i_\mu\omega\left(c_\mu\omega(v)\right)$

**by theorem (2.1.1)** and **(2.1.2),** so that $\omega_\mu - sso \subset \omega_\mu - so$. and

$i_\mu\omega\left(c_\mu\omega(v)\right) \subset i_\mu\omega\left(c_\mu\omega(v)\right) \cup c_\mu\omega\left(i_\mu\omega(v)\right)$

implying $\omega_\mu - so \subset \omega_\mu - \gamma o$. □

**Remark 3.1.**

The conversely of the above diagram in **proposition 3.1** is no true in general, so we give the following example.

**Example 3.1.**

Let $X = \{a, b, c, d\}$, $\omega = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}\}$, $A = \{b, d\}$ $\mu = \{a, b, d\}$, then $v$ is $\omega_\mu$ − semi open and $\omega_\mu$ −pre open but not $\omega_\mu$ −open set

**Example 3.2.**

Let $X = \{a, b, c, d\}$, $\omega = \{\phi, \{a, b\}, \{c, d\}\}$, $\mu = v = \{b, c\}$. Then $v$ is $\omega_\mu - \gamma o$ but not $\omega_\mu - sso$ set.

**Example 3.3.**

Let $X = \{a, b, c, d\}$, $\omega = \{\phi, \{a, b\}, \{a\}, \{b\}\}$, $\mu = v = \{a, c\}$. Then $v = \mu$. Then $v$ is $\omega_\mu - so$ but not $\omega_\mu - o$ set.

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**Example 3.4.**

Let $X = \{a,b,c,d\}$, $\omega = \{\phi, \{a,b,c\}, \{b,c,d\}, \{a,c,d\}\}$, $\mu = \{a,c,d\}$. Then $v = \{a,d\}$ is $\omega_\mu - po$ but not $\omega_\mu - o$ set, and $\alpha = \{a,c\}$ is $\omega_\mu - o$ but not $\omega_\mu - ro$ set.

**Example 3.5.**

Let $X = \{a,b,c,d,e\}$, $\omega = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}\}$, $\mu = \{a,b,c,d\}$. Then $v = \{b,e\}$ is $\omega_\mu - \gamma o$ but not $\omega_\mu - sso$ set.

**Theorem 3.1.**

Let $\omega_\mu$ be a weak structure of $X$, each of the operation :

$$i_\mu\omega\left(c_\mu\omega(i_\mu\omega)\right),$$

$$c_\mu\omega\left(i_\mu\omega(c_\mu\omega)\right)$$

And $c_\mu\omega(i_\mu\omega)$

is monotonic.

In fact if $v \subset \alpha$ then

$$c_\mu\omega\left(i_\mu\omega(v)\right) \subset c_\mu\omega\left(i_\mu\omega(\alpha)\right)$$

$$i_\mu\omega\left(c_\mu\omega(v)\right) \subset i_\mu\omega\left(c_\mu\omega(\alpha)\right)$$

hence

$$c_\mu\omega\left(i_\mu\omega(v)\right) \cup i_\mu\omega\left(c_\mu\omega(v)\right)$$

$$\subset c_\mu\omega\left(i_\mu\omega(\alpha)\right) \cup i_\mu\omega\left(c_\mu\omega(\alpha)\right)$$

**Remark 3.2.**

(1) An $\omega_\mu - o$ set and an $\omega - o$ set are independent concepts.
(2) An $\omega_\mu - so$ set and an $\omega - so$ set are independent concepts.
(3) An $\omega_\mu - po$ set and an $\omega - po$ set are independent concepts.
(4) An $\omega_\mu - sso$ set and an $\omega - sso$ set are independent concepts.
(5) An $\omega_\mu - spo$ set and an $\omega - spo$ set are independent concepts.

**Example 3.6.**

Let $X = \{a,b,c,d\}$, $\omega = \{\phi, \{a\}, \{a,b,c\}, \{c,d\}\}$, $\mu = \{b,c,d\}$, $v = \{a\}$ $\beta = \{b,c\}$, $\alpha = \{c,d\}$, and $\theta = \{a,b,d\}$.

Then $v$ is $\omega - so$ but not $\omega_\mu - so$ set, $\beta$ is $\omega_\mu - o$ but not $\omega - o$ set, $\omega_\mu - so$ but not $\omega - so$ set, $\alpha$ is $\omega - o$ but not $\omega_\mu - so$ set, $\theta$ is $\omega - po(\omega - spo)$ but not $\omega_\mu - po(\omega_\mu - spo)$ set.

**Theorem 3.2.**

Let $(X,\mu)$ be $\omega S$ structure, $\mu \subset X$, and $v \in \omega_\mu S$. Then the following are equivalent:

(i) $v$ is an $\omega_\mu -$ semiclosed set;
(ii) $\mu - v$ is an $\omega_\mu -$ semi open set;
(iii) $i_\mu\omega\left(c_\mu\omega(v)\right) \subseteq v$;
(iii) $c_\mu\omega\left(i_\mu\omega(\mu - v)\right) \supseteq \mu - v$.

**Proof.** It is obvious.

Of course, the above results were contained in the literature (e.g. in [3] ) in the case when the $\omega_\mu S$ is a generalized topology or a minimal structure.

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