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Magic Squares with Pell and Pell-Lucas Entries

KEYWORDS: Pell and Pell-Lucas Entries, Magic Square.

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ABSTRACT *In this paper, I have proved that there is no 2x2 magic square with distinct Pell entries as well as distinct Pell-Lucas entries. Then I generalized this result for nxn magic squares that is, "there are no nxn magic squares with distinct Pell and Pell-Lucas entries where n≥2".*

1. Introduction:

1.1. Magic Square [1]

A magic square is a square array of distinct positive integers such that the sum of the numbers along each row, column and diagonal is constant (say k). This constant k is called the magic constant of the magic square. The oldest known magic square is the Chinese magic square, *lo-shu* shown in figure 1. Lo-shu's magic constant is 15.

4	9	2
3	5	7
8	1	6

Figure 1

1.2. Pell and Pell-Lucas Numbers [2]

Pell and Pell-Lucas numbers are respectively defined as

$$\begin{cases} P_n = 2P_{n-1} + P_{n-2}, & P_0 = 0,\ P_1 = 1, \\ Q_n = 2Q_{n-1} + Q_{n-2}, & Q_0 = 2,\ Q_1 = 2. \end{cases}$$

2. Is there any Magic Square with only Pell Entries?

Answer to this question is – NO. Let us confirm this by contradiction. Suppose there is a 2×2 magic square with distinct Pell entries as figure 2 shows.

a	b
c	d

Figure 2

Then $a+b=a+c$, so $b=c$, which is a contradiction. Thus there is no 2×2 magic square with distinct Pell entries.

Now this result can be generalized as given below:

Theorem: There are no $n\times n$ magic squares with distinct Pell entries where $n \geq 2$.

Proof: Let $P_{i_1}, P_{i_2}, \ldots, P_{i_n}$; $P_{j_1}, P_{j_2}, \ldots, P_{j_n}$ and $P_{k_1}, P_{k_2}, \ldots, P_{k_n}$ denote the elements of the first three columns of an $n\times n$ magic squares with magic constant S, as shown in figure 3.

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P_{i_1}	P_{j_1}	P_{k_1}	……..
P_{i_2}	P_{j_2}	P_{k_2}	……..
. . .	. . .	. . .	……… ……… ………
P_{i_n}	P_{j_n}	P_{k_n}	………

Figure 3

Since,

$$P_{i_1} + P_{i_2} + \ldots + P_{i_n} = P_{j_1} + P_{j_2} + \ldots + P_{j_n} =$$

$$P_{k_1} + P_{k_2} + \ldots + P_{k_n} = \text{S (Say)}$$

But they are all distinct, without loss of generality, we can assume that

$P_{i_1} > P_{i_2} > \ldots > P_{i_n}$; $P_{j_1} > P_{j_2} > \ldots > P_{j_n}$ and $P_{k_1} > P_{k_2} > \ldots > P_{k_n}$

a	b
c	d

Again, without loss of generality, we can assume that $P_{i_1} > P_{j_1} > P_{k_1}$

So $P_{i_1} > P_{k_1}$ and $P_{i_1} \ge P_{k_1+1}$.

$$\Rightarrow \quad 2P_{i_1} \ge P_{k_1} + P_{k_1+1} \qquad (1)$$

Now, $P_{i_1} + P_{i_2} + \ldots + P_{i_n} > P_{i_1}$

$$\Rightarrow \quad 2(P_{i_1} + P_{i_2} + \ldots + P_{i_n}) > 2P_{i_1}$$

$$\Rightarrow \quad 2S > 2P_{i_1} \ge P_{k_1} + P_{k_{11}+1} \quad \text{Using (1)}$$

Since $P_{k_n} < \ldots < P_{k_2} < P_{k_1}$,

Therefore $P_{k_1} + P_{k_2} + \ldots + P_{k_n} < \sum_{i=1}^{k_1} P_i$

Or $\quad 2(P_{k_1} + P_{k_2} + \ldots + P_{k_n}) < 2\sum_{i=1}^{k_1} P_i$

Or $\quad 2S < 2\sum_{i=1}^{k_1} P_i$

But $2\sum_{i=1}^{n} P_i = (P_n + P_{n+1} - 1)$

$$\therefore 2S < (P_{k_1} + P_{k_1+1} - 1) \qquad (2)$$

Using (1) and (2), we have

$$P_{k_1} + P_{k_1+1} \le 2S < (P_{k_1} + P_{k_1+1} - 1)$$

which is a contradiction. Hence, there are no $n \times n$ magic squares with distinct Pell entries where $n \ge 2$.

3. Is There Any Magic Square with Only Pell-Lucas Entries?

Answer to this question is also – NO. Again let us confirm this by contradiction. Suppose there is a 2×2 magic square with distinct Pell-Lucas entries as figure 2 shows. Then $a + b = a + c$, so $b = c$, which is a contradiction. Thus there is no 2×2 magic square with distinct Pell entries.

Now this result can also be generalized as given below:

Theorem: There are no $n \times n$ magic squares with distinct Pell-Lucas entries where $n \ge 2$.

Proof: Let Q_{i_1}, Q_{i_2}, ….., Q_{i_n} ; Q_{j_1}, Q_{j_2}, ….., Q_{j_n} and Q_{k_1}, Q_{k_2}, ….., Q_{k_n} denote the elements of the first three columns of an $n \times n$ magic

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squares with magic constant S , as shown below.

Q_{i_1}	Q_{j_1}	Q_{k_1}	……..
Q_{i_2}	Q_{j_2}	Q_{k_2}	……..
.	.	.	……...
.	.	.	……...
.	.	.	……...
Q_{i_n}	Q_{j_n}	Q_{k_n}	……...

Figure 4

But they are all distinct, without loss of generality, we can assume that

$Q_{i_1} > Q_{i_2} > \ldots > Q_{i_n}$; $Q_{j_1} > Q_{j_2} > \ldots > Q_{j_n}$

and $Q_{k_1} > Q_{k_2} > \ldots > Q_{k_n}$

Again, without loss of generality, we can assume that $Q_{i_1} > Q_{j_1} > Q_{k_1}$

So $Q_{i_1} > Q_{k_1}$ and $Q_{i_1} \geq Q_{k_1+1}$.

$$\Rightarrow \quad 2Q_{i_1} \geq Q_{k_1} + Q_{k_1+1} \qquad (3)$$

Now, $Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n} > Q_{i_1}$

$$\Rightarrow \quad 2(Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n}) > 2Q_{i_1}$$

$$\Rightarrow \quad 2S > 2Q_{i_1} \geq Q_{k_1} + Q_{k_1+1} \text{ Using (3)}$$

Since $Q_{k_n} < \ldots\ldots < Q_{k_2} < Q_{k_1}$,

Therefore, $Q_{k_1} + Q_{k_2} + \ldots + Q_{k_n} < \sum_{i=1}^{k_1} Q_i$

Or $2(Q_{k_1} + Q_{k_2} + \ldots + Q_{k_n}) < 2\sum_{i=1}^{k_1} Q_i$

Or $2S < 2\sum_{i=1}^{k_1} Q_i$

But $2\sum_{i=1}^{n} Q_i = (Q_n + Q_{n+1} - 4)$

$$\therefore \quad 2S < (Q_n + Q_{n+1} - 4) \qquad (4)$$

Using (3) and (4), we have

$$Q_{k_1} + Q_{k_{11}+1} \leq 2S < (Q_n + Q_{n+1} - 4)$$

which is a contradiction. Hence, there are no $n \times n$ magic squares with distinct Pell-Lucas entries where $n \geq 2$.

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