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Generalised Thermoelastic Infinite Medium With A Spherical Cavity Subject to Harmonically Varying Heat

KEYWORDS: Generalized Thermoelasticity, Eigen value, Spherical cavity, Harmonically Heat.

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ABSTRACT *Exact expressions for the temperature distribution ,stress and displacement component are obtained in Laplace transform domain in the case of a infinite medium with a spherical cavity by using eigen value approach in the context of the theory of thermoelasticity with two relaxation time parameters. The surface of the spherical cavity is stress free and subjected to harmonically varying heat. A numerical approach is implemented for the inversion of Laplace transform in order to obtain the solution in physical domain. Finally numerical computations of the stress and temperature have been done and represented graphically.*

INTRODUCTION

The classical theory of thermoelasticity is based on Fourier's law of heat conduction which predicts an infinite speed of propagation of heat.This is physically impossible and many theories have been proposed to eliminate this paradox. Lord and Shulman [7] employed a modified version of the Fourier's law of heat conduction and deduced a thermoelasticity known as Generalized theory of thermoelasticity with one relaxation time parameter. Green and Lindsay [5] presented a theory of thermoelasticity with certain special features that contrast with the prevoius theory having a thermal relaxation time parameter. In this theory Fourier law of heat conduction is unchanged whereas the classical energy equation and strees – strain – temperature relations are modified. Two relaxation time parameters appear in the governing equations in place of one relaxation time paramerter in Lord and Shulman's theory. In both the theories the conventional Fourier law of heat conduction has been modified to a hyperbolic type of equation which along with the equation of motion of thermoelasticity(which are hyperbolic type) are considered for the solution of the problem. Both the theories ensure finite speed of propagation of waves and eliminate automatically the paradox of infinite speed of propagation inherent in both the uncoupled and coupled theories of thermoelasticity, vide, Chandrasekharaiah et al [3]. Using the Green-Lindsay's theory, Lahiri and Kar [6] considered a problem of Generalized thermoelastic interactions in an unbounded body with spherical cavity.

In dealing with coupled or generalized thermoelastic problems, the solution procedure is usually to choose a suitable thermoelastic potential function, but this approach has certain limitations as discussed in Bahar and Hetnarski [1]. Here we prefer to adopt the eigenvalue approach of Das et al [4] for the solution of the such type of problem. In this paper we consider the thermoelastic infinite medium with a spherical cavity within the context of the theory of thermoelasticity with two relaxation times. The medium descibed above is considered to be quiescent and the surface of the cavity is stress free and subjected to hamonically varying heat. Laplace transform have used in the basic equations of thermoelasticity and finally the resulting equations are written in the form of a Vector-Matrix differential equation which is then solved by eigenvalue approach. Finally numerical computations of the stresses and temperature have been made and presented graphically (for different values of time t and angular fequencies).

BASIC EQUATIONS AND FORMULATION

Let us consider a perfectly conducting infinite solid with a spherical cavity occupies the region $a \le r < \infty$ of an isotropic homogeneous medium and analyse the thermoelastic interactions that are spherically symmetric. Then the displacement components have the form $u_r = u(r,t)$, $u_\theta = u_\phi = 0$ and the three principal stresses in the radial , cross radial and transverse directions are $\sigma_{rr}, \sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ respectively. The medium describe above is considered to be quicent and surface of the cavity is subjected to harmonically heat and stress free described mathematically as follows:

$$F(t) = \theta(r,t)\,|_{r=a} = \theta_0 \cos \omega t, \tag{1}$$

$$\sigma_{rr}(a,t) = 0, \tag{2}$$

Where θ_0 is constant and ω is the angular frequency of the thermal vibration. Thus $F(t)$ is a thermal shock for $\omega = 0$.

In the case of spherical symmetry the equation of motion in the radial direction and the energy equation for the Green-Lindsay theory in the absence of external forces or heat sources inside the region are given by Green et al [5]

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial e}{\partial r} - \gamma\left(\frac{\partial T}{\partial r} + v_0 \frac{\partial^2 T}{\partial r \partial t}\right) \tag{3}$$

and

$$k\nabla^2 T = \rho c_E \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2}\right) + \gamma T_0 \frac{\partial e}{\partial t} \tag{4}$$

where e is the cubical dilatation given by

$$e = \frac{\partial u}{\partial r} + \frac{2u}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) \tag{5}$$

and ∇^2 is the one-dimensional Laplace operator in spherical polar coordinates and given by $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$. The constitutive relation for the stresses can be written as [5]

$$\sigma = \sigma_{rr} = \lambda e + 2\mu \frac{\partial u}{\partial r} - \gamma\left(T - T_0 + \nu_0 \frac{\partial T}{\partial t}\right) \tag{6}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \lambda e + 2\mu \frac{u}{r} - \gamma\left(T - T_0 + \nu_0 \frac{\partial T}{\partial t}\right) \tag{7}$$

$$\sigma_{r\theta} = \sigma_{r\phi} = \sigma_{\theta\phi} = 0 \tag{8}$$

After introducing the non-dimensional variables

$$r^* = V\eta r,\quad t^* = V^2\eta t,\quad \theta = \frac{T' - T_0}{T_0},$$
$$u^* = V\eta u,\quad \tau^* = V^2\eta\tau_0,\quad \nu^* = v^2\eta\nu_0,$$
$$\sigma_{ij}^* = \frac{\sigma_{ij}}{\mu},$$

equations (3), (4), (6) and (7) take the following form , where the asterisks are dropped for convenience,

$$\frac{\partial e}{\partial r} - c\frac{\partial}{\partial r}\left(\theta + \nu\frac{\partial \theta}{\partial t}\right) = \frac{\partial^2 u}{\partial t^2} \tag{9}$$

$$\nabla^2\theta = \frac{\partial \theta}{\partial t} + \tau\frac{\partial^2\theta}{\partial t^2} + g\frac{\partial e}{\partial t} \tag{10}$$

$$\sigma = (\beta^2 - 2)e + 2\frac{\partial u}{\partial r} - b\left(\theta + \nu\frac{\partial \theta}{\partial t}\right) \tag{11}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = (\beta^2 - 2)e + 2\frac{u}{r} - b\left(\theta + \nu\frac{\partial \theta}{\partial t}\right) \tag{12}$$

FORMULATION OF A VECTOR-MATRIX DIFFERENTIAL EQUATION

We now apply Laplace transform to the equations (9) we get,

$$L[\overline{u}] = p^2\overline{u} + c(1 + \nu p)\frac{d\overline{\theta}}{dr} \tag{13}$$

and again L.T. to (10), after differentiating w. r. to r and using (13), equation (10) takes the form

$$L\left[\frac{d\overline{\theta}}{dr}\right] = gp^3\overline{u} + p[(1 + \tau p)$$
$$+ gc(1 + \nu p)]\frac{d\overline{\theta}}{dr} \tag{14}$$

$$\text{where}\quad L = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2} \tag{15}$$

Equations (13) and (14) can be written in the form of a Vector-Matrix differential equation as

$$L\widetilde{V} = \widetilde{A}\widetilde{V} \tag{16}$$

where

$$\widetilde{V} = \left[\overline{u}, \frac{d\overline{\theta}}{dr}\right]^T,\quad \widetilde{A} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{17}$$

and

$$c_{11} = p^2,\ c_{12} = c(1 + \nu p),\ c_{21} = gp^3$$

$$c_{22} = p[(1 + \tau p) + gc(1 + \nu p)]$$

SOLUTION OF THE VECTOR-MATRIX EQUATION

To solve the equation (16), we substitute

$$\widetilde{V} = \widetilde{X}(\gamma)\omega(r,\gamma) \tag{18}$$

where γ is a scalar, $\widetilde{X}$ is a vector independent of r and $\omega(r,\gamma)$ is a non-trivial solution of the scalar differential equation

$$L\omega = \gamma^2\omega \tag{19}$$

The solution of the above equation can be written as

$$\omega = \frac{1}{r^2}e^{-\gamma r} + \frac{\gamma}{r}e^{-\gamma r} \tag{20}$$

Using (18) and (19) in (16) and simplifying the result, we obtain the algebraic eigenvalue problem

$$\widetilde{A}\widetilde{X}(\gamma) = \gamma^2\widetilde{X}(\gamma) \tag{21}$$

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where $\widetilde{X}(\gamma)$ is the eigenvector corresponding to the eigenvalue γ^2. The characteristic equation corresponding the matrix $\widetilde{A}$ can be written as

$$\gamma^4 - \gamma^2(c_{11}+c_{22}) + (c_{11}c_{22} - c_{12}c_{21}) = 0 \quad (22)$$

The roots of the characteristic equation (22) are of the form $\gamma = \gamma_1^2$ and $\gamma = \gamma_2^2$

where,

$$\gamma_1^2 + \gamma_2^2 = c_{11} + c_{22} \quad \text{and} \quad \gamma_1^2\gamma_2^2 = c_{11}c_{22} - c_{12}c_{21}. \quad (23)$$

The eigenvectors $X(\gamma_j^2)$, j= 1, 2 corresponding to the eigenvalues γ_j^2, j = 1, 2 can be calculated as

$$\widetilde{X}_j(\gamma_j) = \begin{bmatrix} X_1(\gamma_j^2) \\ X_2(\gamma_j^2) \end{bmatrix} = \begin{bmatrix} -c_{12} \\ c_{11} - \gamma_j^2 \end{bmatrix}_{j=1,2} \quad (24)$$

So, the solution of (18) can be written as

$$\widetilde{V}(r,p) = A\widetilde{X}(\gamma_1^2)\left(\frac{1}{r^2}e^{-\gamma_1 r} + \frac{\gamma_1}{r}e^{-\gamma_1 r}\right)$$

$$+ B\widetilde{X}(\gamma_2^2)\left(\frac{1}{r^2}e^{-\gamma_2 r} + \frac{\gamma_2}{r}e^{-\gamma_2 r}\right). \quad (25)$$

where the constants A and B are to be determined from the boundary conditions. Taking Laplace transform of (11) and using (25) we get

$$\overline{\sigma}(r,p) = AM_1 e^{-\gamma_1 r} + BM_2 e^{-\gamma_2 r} \quad (26)$$

Where

$$M_i = \beta^2 c_{12}\frac{\gamma_i^2}{r} + 4c_{12}\left(\frac{\gamma_i}{r} + \frac{1}{r^3}\right)$$

$$+ b(1+\nu p)\frac{c_{11} - \gamma_i^2}{r}, \; i = 1, 2.$$

After solving the problem, the expressions for the quantities u, θ and σ in Laplace transform domain may be written in the following form :

$$\overline{u}(r,p) = -\frac{acC_1\overline{F}(p)}{r^2\Psi(p)}[C_2\Phi(a,\gamma_2)e^{-\gamma_1(r-a)}$$

$$-C_3\Phi(a,\gamma_1)e^{-\gamma_2(r-a)}] \quad (27)$$

$$\overline{\theta}(r,p) = \frac{a\overline{F}(p)}{\Psi(p)}[(\gamma_1^2 - p^2)\Phi(a,\gamma_2)e^{-\gamma_1(r-a)}$$

$$-(\gamma_2^2 - p^2)\Phi(a,\gamma_1)e^{-\gamma_2(r-a)}] \quad (28)$$

$$\overline{\sigma}(r,p) = \frac{acC_1\overline{F}(p)}{r^3\Psi(p)}[\Phi(r,\gamma_1)\Phi(a,\gamma_2)e^{-\gamma_1(r-a)}$$

$$-\Phi(r,\gamma_2)\Phi(a,\gamma_1)e^{-\gamma_2(r-a)}] \quad (29)$$

where

$$C_1 = 1 + \nu p, \; C_2 = 1 + \gamma_1 r, \; C_3 = 1 + \gamma_2 r$$

$$\Phi(R,\gamma) = \beta^2 p^2 R^2 + 4\gamma R + 4$$

$$\Psi(p) = (\gamma_1 - \gamma_2)[(\beta^2 p^2 a^2 + 4)(\gamma_1 + \gamma_2) + 4a(p^2 + \gamma_1\gamma_2)]$$

and $\overline{F}(p)$ is the Laplace transform of $F(t)$.

Taking the Laplace transform of (1) , we get

$$\overline{F}(p) = \frac{\theta_0 p}{p^2 + \omega^2} \quad (30)$$

NUMERICAL SOLUTION

The Laplace inversion of the expressions for the displacement ,temperature and stress in the physical domain are very complex and we prefer to develop an efficient computer software for the purpose of inversion of Laplace transforms. The inversion of Laplace transform is followed by the method of Bellman [2] and choose a time span by nine values of time t_i i = 1 to 9 at which u, θ and σ have been determined from the negative of the logarithms of the roots of shifted Legendre polynomial of degree nine.

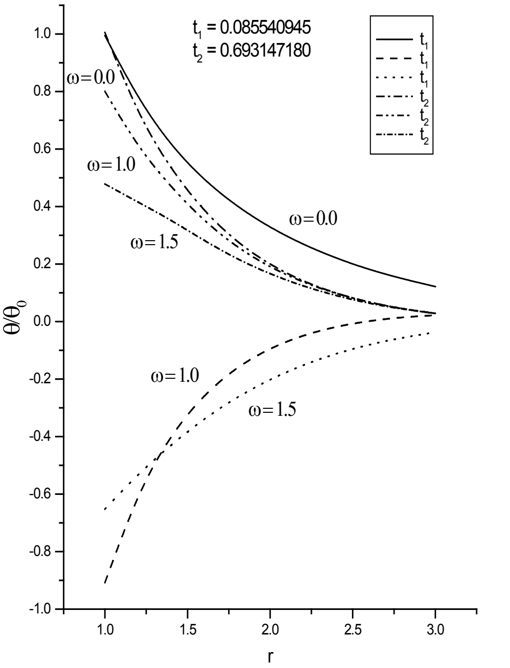

Fig. 1 : Distribution of temperature for different values of omega & time

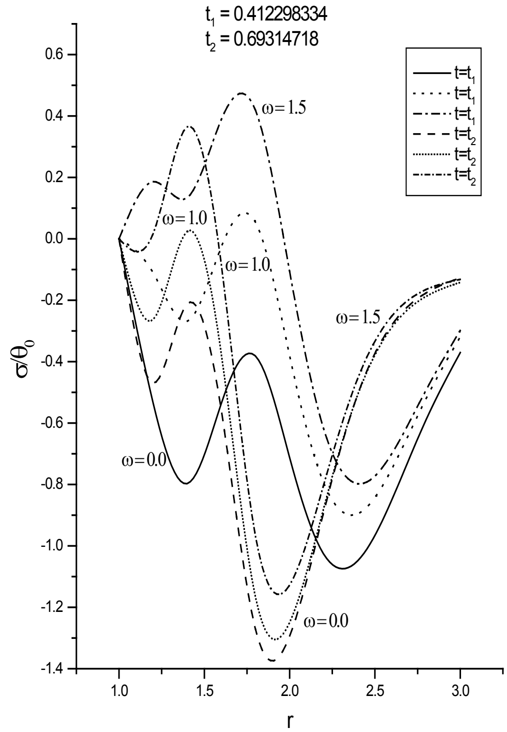

Fig. 3 : Distribution of stresses for different values omega & time

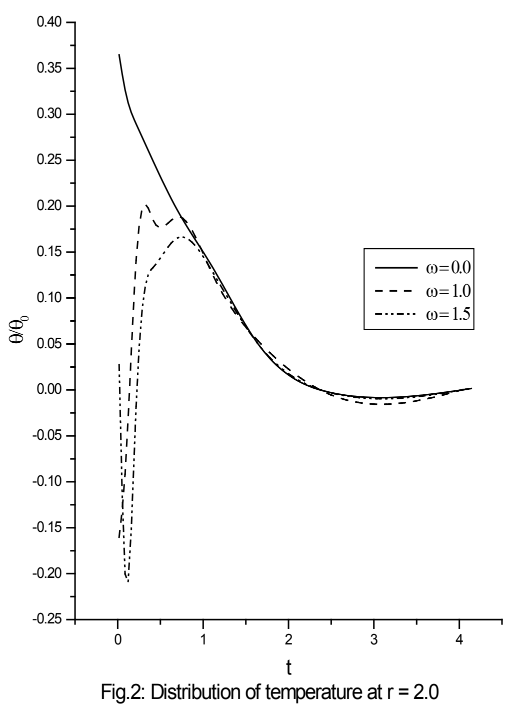

Fig.2: Distribution of temperature at r = 2.0

NUMERICAL RESULTS AND ANALYSIS

The copper material was chosen for the purpose of numerical calculations, vide, Sherief et al. [8] for which
$\varepsilon = 0.0168$, $\beta^2 = 3.5$, and $\tau = \nu = 0.025$
Also we take $a = 1.0$, $\omega = 0, 1.0, 1.5$. The results for temperature and stresses are shown in figures 1 to 4 respectively with wide range of non dimensional distance r from 1.0 to 3.0 and non dimensional time t = 0.086, 0.412 & 0.693 with different values angular frequency of thermal vibration ω = (0.0, 1.0, 1.5) . We can see the significant effect of angular frequency of thermal vibration ω on all the studied field. It is observed from the Figures 1 to 4 that

1. the absolute value of temperature and stresses decrease as value of r, t & ω increase;
2. The value of temperature almost tends to zero when $r > 3$;
3. The value of stresses almost tend to zero when $t > 4$;

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4. Temperature and stresses attain its maximum value (absolute) when $\omega = 0.0$ i.e. the case of thermal shock problem.

CONCLUSION

We considered a perfectly conducting elastic isotropic homogeneous infinite body with spherical cavity in the context of generalized thermoelasticity theory (G-L model). The effect of the angular frequency of thermal vibration on the studied fields are very significant.

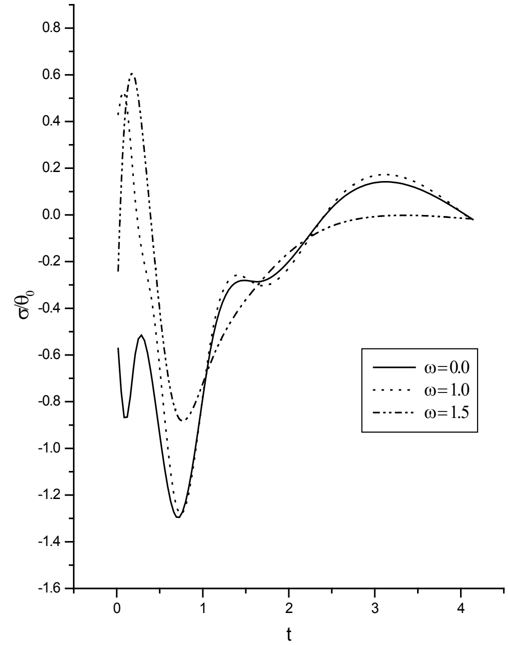

fig 4 : Distribution of stresses at r = 2.0

NOMENCLATURE

(r, θ, ϕ)	(radial coordinate, colatitude, longitude)
λ , μ	Lamé's constants
ρ	density
β	$= \sqrt{\frac{(\lambda+2\mu)}{\mu}}$
V	= speed of propagation of isothermal longitudinal waves $= \sqrt{\frac{(\lambda+2\mu)}{\rho}}$
α_t	Coefficient of linear thermal expansion
γ	$= (3\lambda + 2\mu)\alpha_t$
σ_{ij}	components of stress tensor
u	component of displacement in the radial direction
c_E	Specific heat at constant strain
k	thermal conductivity
η	$\frac{\rho\ c_E}{k}$
t	time
T	absolute temperature
T_0	reference temperature chosen that $\lvert\frac{(T-T_0)}{T_0}\rvert < 1$
θ	$= T - T_0$;Increment of the dynamical temperature
b	$= \frac{\gamma T_0}{\mu}$
c	$= \frac{b}{\beta^2}$
g	$= \frac{\gamma}{\rho\ c_E}$
τ_0	relaxation parameter
q	the heat flux normal to the surface of the cavity

REFERENCE [1] Bahar,L.Y.& Hetnarski,R.B., Connection between the thermoelastic potential and the state space formulation of thermoelasticity, Jour. of Thermal Stresses, 2 (1979), 283-290. | [2] Bellman,R.,Kalaba,R.E. & Lockett,Jo.Ann., Numerical inversion of the Laplace transform , Amer., Elsevier Pub.Co., New York(1966). | [3] Chandrasekharaiah, D. S. & Narasimha Murthy,H., Thermoelastic interactions in an unbounded body with a spherical cavity, Jour. of Thermal stresses , 16 (1993), 55-70. | [4] Das, N. C. and Lahiri, A., Thermoelastic interactions due to prescribed pressure inside a spherical cavity in an unbounded medium, Indian J. Pure Appl. Math., 31(1) (2000), 19-32. | [5] Green, A. E. and Lindsay,K.A., Thermoelasticity, J. Elasticity, Vol. 2(1972), 1-7. | [6] Lahiri,A. and Kar, T.K. , Eigenvalue approach to generalized thermoelastic interactions in an unbounded body with a spherical cavity, IJAME., 12(4) (2007), 1027 - 1037. | [7] Lord,H.W. and Shulman,Y., A generalized dynamical theory of thermoelasticity, J. Mech. Phys. Solids, 15 (1967), 299-309. |