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# Convergence and Stability For Ishikawa Iterative Scheme in Convex Metric Spaces

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**ABSTRACT** *In this paper, we prove the convergence and stability results for two step Ishikawa iterative scheme in convex metric spaces for a self-mapping satisfying general quasi-contractive condition.*

## INTRODUCTION AND PRELIMINARIES

Convergence of fixed points iterative schemes in convex metric spaces has been the subject of research in fixed point theory for some time now [see [4-6,12] and several references therein].

In 1970, Takahashi [12] introduced the concept of convexity in metric space $(X,d)$ as follows:

**Definition 1.1** A map $W: X^2 \times [0,1] \to X$ is a convex structure in $X$ if

$$d(u, W(x,y,\lambda)) \le \lambda d(u,x) + (1-\lambda) d(u,y)$$

for all $x, y, u \in X$ and $\lambda \in [0,1]$. A metric space $(X,d)$ together with a convex structure W is known as convex metric space and is denoted by $(X,d,W)$. A nonempty subset $C$ of a convex metric space is convex if $W(x,y,\lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$.

The stability theory of fixed point iteration schemes has been systematically studied by many authors using various contractive conditions[1-3, 6-11]. In 2011, Olatinwo [6] defined the concept of T-stability in convex metric space setting:

**Definition 1.2:** Let $(X,d,W)$ be a convex metric space and $T: X \to X$ a self mapping.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative scheme involving $T$ which is defined by

$$x_{n+1} = f_{T,\alpha_n}^{x_n},\ n = 0,1,2.... \quad (1.1)$$

where $x_0 \in X$ is the initial approximation and $f_{T,\alpha_n}^{x_n}$ is some function having convex structure such that $\alpha_n \in [0,1]$. Suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set $\varepsilon_n = d(y_{n+1}, f_{T,\alpha_n}^{y_n}), (n = 0,1,2....)$. Then, the iterative scheme (1.1) is said to be T-stable with respect to $T$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$, implies $\lim_{n\to\infty} y_n = p$.

Now we recapitulate some of the iterative schemes in terms of convex structure as follows:

Let $(X,d,W)$ be a convex metric space and $T: X \to X$ be a self map of X. For $x_0 \in X$,

Picard iterative scheme[11]:

$$x_{n+1} = Tx_n,\ n = 0,1,2.... \quad (1.2)$$

Mann iterative scheme[4,13]:

$$x_{n+1} = W(x_n, Tx_n, \alpha_n),\ n = 0,1,2.... \quad (1.3)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1].

Ishikawa iterative scheme[4,5]:

$$x_{n+1} = W(x_n, Ty_n, \alpha_n)$$

$$y_n = W(x_n, Tx_n, \beta_n),\ n = 0,1,2.... \quad (1.4)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

Osilike and Udomene [9] defined a new general definition of quasi contractive operator as follows:

$$d(Tx,Ty) \le \delta d(x,y) + L d(x,Tx)$$

$\forall x, y \in X$ and some $L \geq 0, \delta \in [0,1]$.
(1.5)

A more general definition was introduced by Imoru and Olatinwo [3] as follows which we will use in our results: if there exists a constant $0 \leq \delta < 1$ and a monotonically increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \delta d(x, y) + \varphi(d(x, Tx)) \quad .$$
(1.6)

## 1. MAIN RESULTS:

First we give a lemma which is used in our main results.

**Lemma 2.1 [11]** If $\delta$ is a real number such that $0 \leq \delta < 1$, and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n + \varepsilon_n$, $n = 0,1,2.....$; we have $\lim_{n\to\infty} u_n = 0$.

**Theorem 2.2** Let $K$ be a nonempty closed convex subset of a convex metric spaces $X$ and $T : K \to K$ be a mapping satisfying (1.6). Let $\{x_n\}_{n=0}^{\infty}$ be defined through the Ishikawa iterative scheme (1.4) and $x_0 \in X$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of $T$.

**Proof:** Let p be the fixed point of T. Then, from (1.4), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, Ty_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(Ty_n, p) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n \delta d(y_n, p) \end{aligned}$$
(2.1)

Similarly from (1.4), we have the following estimates:

$$\begin{aligned} d(y_n, p) &= d(W(x_n, Tx_n, \beta_n), p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(Tx_n, p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n \delta d(x_n, p) \\ &= (1-\beta_n(1-\delta)) d(x_n, p) \end{aligned}$$
(2.2)

Using (2.1) and (2.2) , we have

$$\begin{aligned} d(x_{n+1}, p) &\leq [1-\alpha_n(1-\delta)] d(x_n, p) \\ &------ \\ &\leq \prod_{k=0}^{n} [1-\alpha_k(1-\delta)] d(x_0, p) \\ &\leq e^{-(1-\delta)\sum_{k=o}^{\infty}\alpha_k} d(x_0, p). \end{aligned}$$
(2.3)

Since $0 \leq \delta < 1$, $\alpha_k \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, so $e^{-(1-\delta)\sum_{k=0}^{\infty}\alpha_k} \to 0$ as $n \to \infty$. Hence, it follows from (2.3) that $\lim_{n\to\infty} d(x_{n+1}, p) = 0$. Therefore $\{x_n\}_{n=0}^{\infty}$ converges strongly to $p$. Hence the result.

**Theorem 2.3.** Let $(X,d,W)$ be a complete convex metric space and $T:X\to X$ a mapping satisfying contractive condition (1.6). Suppose that T has a fixed point p. For $x_0\in X$, let Ishikawa iterative scheme $\{x_n\}_{n=0}^{\infty}$ be defined by (1.4), where $\alpha_n,\beta_n\in[0,1]$ such that $0<\alpha\le\alpha_n$. Then the Ishikawa iterative scheme is T-stable.

**Proof.** Suppose that $\{y_n\}_{n=0}^{\infty}\subset X$ is an arbitrary sequence in X and define $\varepsilon_n=d(y_{n+1},W(\mathrm{y}_n,Tq_n,\alpha_n))$ where $q_n=W(y_n,Ty_n,\gamma_n)$.

Let $\lim_{n\to\infty}\varepsilon_n=0$. Then by using contractive condition (1.6), we shall prove that $\lim_{n\to\infty}y_n=p$.

Using triangle inequality we have

$$d(y_{n+1},p)\le d(y_{n+1},W(\mathrm{y}_n,Tq_n,\alpha_n))+d(W(\mathrm{y}_n,Tq_n,\alpha_n),p)$$

$$\le\varepsilon_n+(1-\alpha_n)d(\mathrm{y}_n,p)+\alpha_n d(Tq_n,Tp)$$

$$\le\varepsilon_n+(1-\alpha_n)d(\mathrm{y}_n,p)+\alpha_n\delta d(\mathrm{q}_n,p)$$

(2.4)

For the estimate of $d(\mathrm{q}_n,p)$ in (2.4), we get

$$d(\mathrm{q}_n,p)=d(W(\mathrm{y}_n,Ty_n,\beta_n),p)$$

$$\le(1-\beta_n)d(\mathrm{y}_n,p)+\beta_n d(Ty_n,Tp)$$

$$\le(1-\beta_n)d(\mathrm{y}_n,p)+\beta_n\delta d(y_n,p)$$

(2.5)

Substituting (2.5) into (2.4), we get

$$d(y_{n+1},p)\le\varepsilon_n+(1-\alpha_n)\,\mathrm{d}(\mathrm{y}_n,\mathrm{p})$$
$$+\alpha_n\delta[(1-\beta_n)\,\mathrm{d}(\mathrm{y}_n,\mathrm{p})+\beta_n\delta d(y_n,p)]$$

$$=\varepsilon_n+[(1-(1-\delta)\alpha_n-\alpha_n\beta_n\delta(1-\delta)]\,\mathrm{d}(\mathrm{y}_n,\mathrm{p})$$

(2.6)

Observe that, $0\le(1-\alpha(1-\delta))<1$

Therefore, taking the limit as $n\to\infty$ on both sides of the inequality (2.6), and using Lemma 2.1, we get $\lim_{n\to\infty}d(y_n,p)=0$, that is $\lim_{n\to\infty}y_n=p$.

Conversely, let $\lim_{n\to\infty}y_n=p$. So,

$$\varepsilon_n=d(y_{n+1},W(\mathrm{y}_n,Tq_n,\alpha_n))$$

$$\le d(y_{n+1},p)+d(W(\mathrm{y}_n,Tq_n,\alpha_n),p)$$

$$\le d(y_{n+1},p)+(1-\alpha_n)d(\mathrm{y}_n,p)+\alpha_n d(Tq_n,Tp).$$

$$\le d(y_{n+1},p)+(1-\alpha(1-\delta))\,\mathrm{d}(\mathrm{y}_n,\mathrm{p})\to 0 \text{ as } n\to\infty.$$

Hence the proof.

**Remark 2.4.** As Mann iterative scheme is a special case of Ishikawa iterative scheme, convergence and stability results for Mann iterative scheme in convex metric spaces can be proved similar to Theorem 2.2 and Theorem 2.3 .

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